1. Compute the minimal polynomial of $i + \sqrt{2}$ over $\mathbb{Q}$.

2. Suppose $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 35$. Show that $\mathbb{Q}(\alpha^3) = \mathbb{Q}(\alpha)$.

3. Let $K = \mathbb{Q}(\zeta_{16})$ and let $G = \{\sigma_1, \sigma_3, \sigma_5, \ldots, \sigma_{15}\}$ be the Galois group of K over $\mathbb{Q}$, where σ_i maps ζ_{16} to ζ_{16}^i. For each of the following subgroups H of G, write down:
(i) the fixed field K^H, (ii) its degree $[K^H : \mathbb{Q}]$.

   (a) G

   (b) $< \sigma_1 >$.

   (c) $< \sigma_3 >$.

   (d) $< \sigma_5 >$.

   (e) $< \sigma_7 >$.

   (f) $< \sigma_9 >$.

   (g) $< \sigma_{15} >$.

   (h) Which of the field(s) in questions (a)–(g) contains i?

   (i) Which of the field(s) in questions (a)–(g) is contained in $\mathbb{R}$?

4. Let $K = \mathbb{Q}(i, \sqrt[4]{3})$.

   (a) What is the Galois group G of K over $\mathbb{Q}$?

   (b) Give a subgroup H of G whose fixed field is:

      i. $\mathbb{Q}(i)$.

      ii. $\mathbb{Q}(\sqrt[4]{3})$.

      iii. $\mathbb{Q}(i\sqrt[4]{3})$.

5. Let $K := \mathbb{Q}(\zeta_{16})$ and let G be its Galois group. For each of the following subfields E, write down an explicit group $H \leq G$ such that E is the fixed field of H. You do not need to explain your answers for (a)–(f).

   (a) $E_1 := K$

   (b) $E_2 := \mathbb{Q}$

   (c) $E_3 := K \bigcap \mathbb{R}$

   (d) $E_4 := \mathbb{Q}(\zeta_{16} + \zeta_{16}^7)$

   (e) $E_5 := \mathbb{Q}(\zeta_8)$

   (f) $E_6 := \mathbb{Q}(\zeta_4)$

   (g) $E_7 :=$ The intersection of E_3 and E_5.

   (h) What is $[E_7 : \mathbb{Q}]$?

   (i) Is $E_7 \subseteq E_4$?

6. Let $K = \mathbb{Q}(i, \sqrt[4]{3})$ and let $G =< \tau, \sigma >$ where τ is complex conjugation, $\tau : i \mapsto -i$, and σ sends i to i and $\sqrt[4]{3}$ to $i\sqrt[4]{3}$. Let $h = \tau\sigma^2$ and $H =< h >$. What is K^H?

7. Suppose that $K/\mathbb{Q}$ is Galois and that its Galois group G is a simple group. Suppose that E is a proper subfield, i.e. $\mathbb{Q} \subsetneq E \subsetneq K$. Show that $E/\mathbb{Q}$ is not Galois.

8. Suppose that $K/\mathbb{Q}$ is Galois with group G. Suppose that $\alpha \in K$ and that $\sigma \in Z(G)$, the center of G. Show that $\sigma(\alpha) \in \mathbb{Q}(\alpha)$.

   Hint: Apply Galois correspondence to $E := \mathbb{Q}(\alpha)$ and......

9. Let E be a subfield of $\mathbb{Q}(\zeta_{17})$, not equal to $\mathbb{Q}(\zeta_{17})$. Show that $E \subset \mathbb{R}$.

10. Let $K := \mathbb{Q}(\zeta_{16}) \bigcap \mathbb{R}$. Show that K is Galois over $\mathbb{Q}$, and give its Galois group.

11. Let $K = \mathbb{Q}(\zeta_{13})$.

    (a) Is K Galois over $\mathbb{Q}$?

    (b) How many subfields does K have. (Include K and $\mathbb{Q}$ in your count).

    (c) How many of subfields of K are inside $\mathbb{R}$?

    (d) Does there exist an element $a \in K$ with $a \notin \mathbb{R}$ and $\mathbb{Q}(a) \neq K$? If so, then write down an example of such a.

12. Let $\zeta = e^{2\pi i/31}$ be a primitive 31'th root of unity, and let

$$\alpha = \zeta + \zeta^2 + \zeta^4 + \zeta^8 + \zeta^{16}.$$

    Let $K = \mathbb{Q}(\zeta)$ and $E = \mathbb{Q}(\alpha)$. Let G be the Galois group of K over $\mathbb{Q}$.

    (a) Write down the group G and the order of G.

    (b) Explain why E must be Galois over $\mathbb{Q}$.

    (c) Prove that $[E : \mathbb{Q}] \leq 6$ (hint: Write down a subgroup $H \leq G$ such that α is in the fixed field of H).

    (d) Prove that $[E : \mathbb{Q}] = 6$.

    (e) Let $\beta = \zeta^3 + \zeta^6 + \zeta^{12} + \zeta^{24} + \zeta^{48}$ (note: $\zeta^{48} = \zeta^{17}$). Prove that $\beta \in E$.