GRV II, sample questions.

1. (10 points). Let R be a commutative ring with identity. Show that there is a field K for which there exists an onto homomorphism from R to K. (3 points bonus: is this provable without the axiom of choice?)

2. Let $R = \mathbb{R}[x]/(f(x))$ where $f(x)$ is a non-constant polynomial in $\mathbb{R}[x]$.

   (a) (15 points). Prove that every ideal in R is principal (note: do not write that R is a PID, because if $f(x)$ is reducible then R fails the letter D, domain, in PID).

   (b) (5 points). Let $f(x) = x^3 + x$. For this case, list explicitly all ideals of R (no proof is necessary for this question, but your proof for question (a) can be very helpful here. Note: list *all* ideals, including the trivial ones).

   (c) (10 points). Again $f(x) = x^3 + x$. Give an isomorphism from R to the product $F_1 \times F_2$ for some fields F_1, F_2.

   (d) (5 points). Again $f(x) = x^3 + x$. The equation $e^2 = e$, how many solutions does this equation have in R?

3. Let $R = \mathbb{Z}[\sqrt{-7}]$.

   (a) (5 points). Show that $1 + \sqrt{-7}$ and 2 are irreducible in R.
   Hint: Introduce the Norm $N(x) := x \cdot \overline{x}$ where $\overline{x}$ is the conjugate of x. Note that $N(xy) = N(x)N(y)$ and if $x = a + b\sqrt{-7}$ then $N(x) = a^2 + 7b^2$. Show that the only units in R are ± 1, there are no elements of Norm 2, and the only elements of Norm 4 are ± 2.

   (b) (5 points). Show explicitly that R is not a UFD by factoring 8 in two non-equivalent ways as a product of irreducible elements.
   Explain why the Norm N is not a Euclidean Norm.

   (c) (10 points). Let $I \neq R$ be an ideal, and assume that $a, b \in R$, $ab \in I$ while $a, b \notin I$ (in other words, I is not prime). Let $J = (I, a)$. Show that $I \subsetneq J \subsetneq R$.
   Hint: the only non-trivial thing to show is that $J \neq R$, which you can show by first showing that $Jb \subseteq I$ while $Rb \not\subseteq I$.

   (d) (10 points). Prove that the ideal (2) is not maximal in R, and that the ideal $(2, 1 + \sqrt{-7})$ is not equal to R.

   (e) (10 points). Prove that the ideal $(2, 1 + \sqrt{-7})$ is not principal (hint: you may use parts (a)+(d) even if you did not prove them).

4. (15 points). Let R be a UFD, and let $f = a_n x^n + \cdots + a_0 x^0 \in R[x]$ with $a_0, a_n \neq 0$. Let K be the field of fractions, and suppose that $x^2 + bx + c \in K[x]$ is a factor of f in $K[x]$. Prove that $a_n \cdot (x^2 + bx + c) \in R[x]$.

5. Let G be a subgroup of S_{10} of order $81 = 3^4$. Let $S = \{1, 2, \ldots, 10\}$. The group S_{10} acts on S, and hence, G acts on S as well. Prove that the action of G on S must have a fix point, i.e., prove that there exist an $s \in S$ such that $ga = a$ for all $g \in G$.

6. Let G_1 and G_2 be subgroups of S_{10} of order 81. Prove that G_1 is isomorphic to G_2.

7. D_{50}, the dihedral group of order 50, how many elements does it have of order:
   1:
   2:
   5:
   10:
   25:
   50:

8. (a) List every abelian group of order 600 (up to isomorphism) (in other words, if $G_1 \cong G_2$ then do not list both).
   (b) List every abelian group of order 64 (up to isomorphism).
   (c) List every abelian group of order 64 that has elements of order 8 but no elements of order 16.
   (d) List every abelian group of order 64 in which the equation $g^2 = e$ has precisely 8 solutions.

9. Let $G = \{ax + b \,|\, a \in \mathbb{R}^*,\ b \in \mathbb{R}\}$.
   So $G = \{\text{non-constant linear functions } \mathbb{R} \to \mathbb{R}\}$, which is a group under composition.
   Notice that in the first definition of G, you have a in a multiplicative group $\mathbb{R}^*$ and b in an additive group $\mathbb{R}$. Can you write G as a semi-direct product of those two groups?
   (if yes, just write down such a semi-direct product (don't forget to include a map from ... to ...). You don't have to prove that your answer is isomorphic to G)

10. Suppose that $d > 1$ and $d|\phi(n)$ where ϕ is the Euler ϕ function ($\phi(n)$ is the number of units in the ring $\mathbb{Z}/(n)$).
    Show that there exists a nonabelian group of order nd.