Sample questions.

1. Let M be a finitely generated module over a PID R. Let $m \in M$. The annihilator of m is the set of all $r \in R$ with $rm = 0$. The annihilator of M is the of all $r \in R$ for which $rm = 0$ for all $m \in M$. Show that there exists $m \in M$ whose annihilator is equal to the annihilator of M.

2. Let M be $\mathbb{Q}[x]$-module, and a $\mathbb{Q}$-vector space of dimension n. Let (f) be the annihilator of M. Show that the degree of f is n if and only if M is a cyclic $\mathbb{Q}[x]$-module.

3. Let $\mathbb{F}_2$ be the field with 2 elements and let $R = \mathbb{F}_2[x]$. Up to isomorphism, how many R-modules exist with precisely 8 elements?

4. For $a, b \in \mathbb{Z}$ let $v = (a, b)$ in the abelian group $\mathbb{Z} \times \mathbb{Z}$. Show that the quotient group $\mathbb{Z} \times \mathbb{Z} / < v >$ is cyclic if and only if $\gcd(a, b) = 1$.

5. Let R be a PID, and let $\phi : R^n \to R^n$ be a homomorphism of R-modules. Show that for some d there is a surjective homomorphism $\phi_2 : R^n \to R^d$ with the same kernel as ϕ.

6. Let R be a PID and let M be a finitely generated R-module. Show that the following are equivalent:

   (a) M is not cyclic.

   (b) There exists a surjective R-module homomorphism

   $$\phi : M \to R/I \times R/I$$

   for some ideal $I \subsetneq R$.

7. Let $f(x)$ and $g(x)$ be two polynomials in $\mathbb{R}[x]$ of degree 3. Suppose that $f'(x) > 0$ and $g'(x) > 0$ for every $x \in \mathbb{R}$. Prove that $\mathbb{R}[x]/(f(x))$ and $\mathbb{R}[x]/(g(x))$ are isomorphic as rings.

8. (These last questions use material that will be covered this week). Let V be an $\mathbb{R}$-vector space of dimension n. Let $\phi : V \to V$ be a linear map for which $\phi^3(v) = \phi^2(v)$ for all $v \in V$. Prove that there exists a basis $b_1, \ldots, b_n$ of V for which $\phi(b_i) \in \{0, b_i, b_{i-1}\}$ for every $i = 1, \ldots, n$.

9. Show that the following three are equivalent:

   (a) There is a non-trivial subspace $0 \neq W \subsetneq F^n$ with $Mw \in W$ for all $w \in W$.

   (b) M is similar to a matrix of the form

   $$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

   for some matrices A, B, C with entries in F (of sizes $m_1 \times m_1$, $m_1 \times m_2$, $m_2 \times m_2$ for some $m_1, m_2 > 0$).

   (c) The characteristic polynomial $f \in F[x]$ of M is reducible in $F[x]$.

   Note: Proving (a) $\implies$ (b) $\implies$ (c) gives you credit for 1 full exercise, and proving (c) $\implies$ (a) is bonus (hint: view F^n as an $F[x]$-module).

10. Let M be an $\mathbb{F}_p[x]$-module of dimension 3 (dimension as $\mathbb{F}_p$-vector space). Suppose that M is not a cyclic module.

    (a) Let $r = x^p - x$ and let $m \in M$. Show that $r^2 m = 0$.

    (b) Must rm also be 0?