Handout Section 2.1, Intro Advanced Math

Let A be a set. We say that A is *countable* when:

(1) The set A is either:

  (a) *Finite.*
  This means that there exists an integer n, and $a_1, \ldots, a_n$, for which $A = \{a_1, \ldots, a_n\}$.
  Note: A is allowed to be the empty set, in that case take $n = 0$. When $n = 0$, you should interpret $\{a_1, \ldots, a_n\}$ as the empty set.

  (b) or: *Countably infinite.*
  This means that there exists a bijection $f : \mathbb{N}^* \to A$. Recall that $\mathbb{N}^*$ denotes $\{1, 2, 3, \ldots\}$.

(2) There exists an injective function $g : A \to \mathbb{N}^*$.

(3) $A = \emptyset$ or there exists an onto function $h : \mathbb{N}^* \to A$.

(4) $A = \emptyset$ or there exists a sequence $a_1, a_2, a_3, \ldots$ such that $A = \{a_1, a_2, a_3, \ldots\}$.

(5) There exists a sequence $a_1, a_2, a_3, \ldots$ such that $A \subseteq \{a_1, a_2, a_3, \ldots\}$.

Conditions (1)–(5) are *equivalent*, so they are either all true (then A is countable) or all false (then A is uncountable).

The main results in Section 2.1 are:

- Theorem 1: a countable union of countable sets is countable. So if you have a countable set A_i, for each i in some countable set I, then the union of these A_i (notation: $\bigcup_{i \in I} A_i$) is again a countable set.
- $\mathbb{Z}$ and $\mathbb{Q}$ are countable but $\mathbb{R}$ is not.

Lets use (1)–(5) to do some exercises of section 2.1.

1. Ex 1. Let A countable and $f : A \to B$ is onto. To prove: B is countable. Looking for the phrase "onto" in conditions (1)–(5), it seems that our best bet is to look for an onto function from $\mathbb{N}^*$ to B.

   Proof: A is countable, so if $A \neq \emptyset$ then according to (3) there exists an onto function $h : \mathbb{N}^* \to A$. Composing this with f gives an onto function $\mathbb{N}^* \to B$. Hence, B satisfies (3) and is thus countable.

2. Ex 2. Let A, B countable. For each $i \in B$, let $A_i := A \times \{i\}$. Then $A \times B$ equals $\bigcup_{i \in B} A_i$. Since B and the A_i are countable, we see that $A \times B$ is countable by Theorem 1.

3. Ex 3. Let A be a countable set. Let $A^n = \{(a_1, \ldots, a_n) | a_i \in A\}$.
(This is the set of all n-tuples over A (an n-tuple is a list with n entries)).

   A^n is the cartesian product of n copies of A. So $A^2 = A \times A$ and $A^3 = A \times A \times A$, etc. By Exercise 2 these are countable.
   Note: A^{n+1} can be viewed as $A^n \times A$. Let B_n be the set $\{\{a_1, \ldots, a_n\} | a_i \in A\}$. The function $A^n \to B_n$ that sends $(a_1, \ldots, a_n)$ to $\{a_1, \ldots, a_n\}$ is onto, and thus B_n is countable by condition (3). Notice that every subset S of A with n elements is an element of B_n. Let $B = B_0 \bigcup B_1 \bigcup B_2 \cdots$. Then every subset S of A with a finite number of elements is an element of B_n for some n, and thus an element of B. But B is a countable set by Theorem 1.

4. Ex 4. There are many ways to do this. For example, we could let A_k be the set of all integers of the form $k \cdot 2^j$ for some $j = 0, 1, 2, \ldots$. Then $\mathbb{N}^* = A_1 \bigcup A_3 \bigcup A_5 \bigcup \cdots$.
   Another answer is this: Let B_1 be the set of all prime numbers, union $\{1\}$ (note: 1 is not a prime). Then for $k > 1$, let B_k be the set of all integers that can be written as a product of k primes. Then $\mathbb{N}^* = B_1 \bigcup B_2 \bigcup B_3 \bigcup \cdots$.

5. Ex 10. A is infinite and B is a finite subset of A. So we can write $B = \{a_1, \ldots, a_n\}$ for some $n \geq 0$, and some $a_i \in A$. Now choose distinct $a_{n+1}, a_{n+2}, \ldots \in C = A - B$. We can do this because C is infinite (note: it does require us to make infinitely many choices, more on that later). Now make the following function $f : A \to C$. If $a = a_i$ for some i, then $f(a) = a_{n+i}$. Otherwise $f(a) = a$. Then f is a bijection from A to C (so A and C have the same cardinality). To summarize: removing (or adding!) finitely many elements from (to) an infinite set does not change its cardinality. That'll come in handy in Ex 3 in section 2.2.