Intro Advanced Math, test 1. **Your name:**

10 1. Let A, B be sets and suppose that $A - B = B$. Then show that A and B are both empty.

Answer: To prove $B = \emptyset$ (i.e. $\forall_x\ x \notin B$) we will show that $x \in B$ leads to a contradiction.
Assume $x \in B$. Then $x \in A - B$ since $A - B = B$, so $x \in A$ and $x \notin B$, contradicting the assumption.
Now that we know $B = \emptyset$ we find $A = A - \emptyset = A - B = B = \emptyset$.

35 (5 EACH) 2. (a) Let f be a function from A to B. Write down the *contrapositive* of this statement:

p: $f(x) = f(y) \Longrightarrow x = y$

Answer: $x \neq y \Longrightarrow f(x) \neq f(y)$.

(b) Write down the *converse* of statement p.

Answer: $x = y \Longrightarrow f(x) = f(y)$.

(c) Is there a statement among your answers for (a),(b) that is true for every function?

Yes, (b) says that f is well-defined. That is true for every function.

(d) Now compute the *negation* of this statement:

q: There exists $b \in B$ such that $b \neq f(a)$ for every $a \in A$.

Answer: $\neg q$ says $\forall_{b \in B} \exists_{a \in A}\ b = f(a)$.

(e) Can you express your answer for $\neg q$ in terms of one of the phrases/definitions you memorized?

$\neg q$ says that f is onto.

(f) Let L be a chain, let S be a subset of L, and consider this statement:

r: For every x in S there exists y in S with $y > x$.

Compute the *negation* of r (Recall that in a chain, the negation of $y > x$ is simply $y \leq x$).

Answer: $\neg r$ says $\exists_{x \in S} \forall_{y \in S}\ y \leq x$.

(g) Can you express your answer for $\neg r$ in terms of one of the phrases we have learned?

Answer: $\neg r$ says: $\exists_{x \in S}$ such that x is an upper bound for S.
An upper bound for S that happens to be in S is called a top element.
So $\neg r$ says that S has a top element.

25 (5 EACH) 3. (a) Give the definition of injective (a.k.a. one to one): $f : A \to B$ is injective when:

5 Answer: (for all $a_1, a_2 \in A$): $f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$.

(b) Give the definition of surjective (a.k.a. onto): $f : A \to B$ is surjective when:

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Answer: $\forall_{b\in B}\exists_{a\in A}\ f(a) = b$.

(c) Suppose that $f : A \to B$ and $g : B \to A$ and (1): $\forall_{a\in A}\ g(f(a)) = a$.

i. Prove that f is injective.

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Answer: Assume $f(a_1) = f(a_2)$. To prove: $a_1 = a_2$.
Apply g to the assumed statement gives: $g(f(a_1)) = g(f(a_2))$.
Applying (1) to the last equation gives $a_1 = a_2$.

ii. Prove that g is surjective.

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Answer: $g : B \to A$ so we have to prove that if $a \in A$ then there exists $b \in B$ with $g(b) = a$.
Proof: Take $b := f(a)$ (then $g(b) = g(f(a))$ equals a by (1)).

iii. If f is surjective then show that g is injective.

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Assume $g(b_1) = g(b_2)$, to prove: $b_1 = b_2$.
Since f is surjective, there are $a_1, a_2 \in A$ with $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Applying g we get $g(b_1) = g(f(a_1)) = a_1$ (last equation used (1)). Likewise $g(b_2) = a_2$. But we assumed $g(b_1) = g(b_2)$ and so $a_1 = a_2$. Then $b_1 = f(a_1) = f(a_2) = b_2$.

4. (a) Let x and y be real numbers: Consider the statement

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$$(\forall_{\epsilon>0}\ x < y + \epsilon) \implies x \le y$$

Write down the *contrapositive* of this statement and simplify your answer so that you have no negation symbol in front of a quantifier.

Answer: $x > y \implies (\exists_{\epsilon>0}\ x \ge y + \epsilon)$.

(b) Can you prove the statement?

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Answer: Assume $x > y$.
Take $\epsilon := x - y$ (then $\epsilon > 0$ and $x \ge y + \epsilon$).

5. Bonus or take-home question: Suppose that A, B, I are sets, and C_i is a set for every $i \in I$. Suppose that $C_i \subseteq B$ for every $i \in I$. Show that

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$$A \setminus B \subseteq \bigcap_{i\in I} A \setminus C_i$$

(Note: $A \setminus B$ is the same as $A - B$)