Intro Advanced Math, test 1.

Your name:

1. Let \( A, B \) be sets and suppose that \( A - B = B \). Then show that \( A \) and \( B \) are both empty.

   Answer: To prove \( B = \emptyset \) (i.e. \( \forall x \ x \notin B \)) we will show that \( x \in B \) leads to a contradiction.
   Assume \( x \in B \). Then \( x \in A - B \) since \( A - B = B \), so \( x \in A \) and \( x \notin B \), contradicting the assumption.
   Now that we know \( B = \emptyset \) we find \( A = A - \emptyset = A - B = B = \emptyset \).

2. (a) Let \( f \) be a function from \( A \) to \( B \). Write down the contrapositive of this statement:

   \[ p: \quad f(x) = f(y) \implies x = y \]
   Answer: \( x \neq y \implies f(x) \neq f(y) \).

   (b) Write down the converse of statement \( p \).
   Answer: \( x = y \implies f(x) = f(y) \).

   (c) Is there a statement among your answers for (a),(b) that is true for every function?
   Yes, (b) says that \( f \) is well-defined. That is true for every function.

   (d) Now compute the negation of this statement:
   \[ q: \quad \text{There exists } b \in B \text{ such that } b \neq f(a) \text{ for every } a \in A \]
   Answer: \( \neg q \) says \( \forall b \in B \exists a \in A \ b \neq f(a) \).

   (e) Can you express your answer for \( \neg q \) in terms of one of the phrases/definitions you memorized?
   \( \neg q \) says \( f \) is onto.

   (f) Let \( L \) be a chain, let \( S \) be a subset of \( L \), and consider this statement:
   \[ r: \quad \text{For every } x \text{ in } S \text{ there exists } y \text{ in } S \text{ with } y > x \]
   Compute the negation of \( r \) (Recall that in a chain, the negation of \( y > x \) is simply \( y \leq x \)).
   Answer: \( \neg r \) says \( \exists x \in S \forall y \in S \ y \leq x \).

   (g) Can you express your answer for \( \neg r \) in terms of one of the phrases we have learned?
   \( \neg r \) says: \( \exists x \in S \) such that \( x \) is an upper bound for \( S \).
   An upper bound for \( S \) that happens to be in \( S \) is called a top element.
   So \( \neg r \) says that \( S \) has a top element.

3. (a) Give the definition of injective (a.k.a. one to one): \( f: A \to B \) is injective when:

   \( f(a_1) = f(a_2) \implies a_1 = a_2 \).

   Answer: (for all \( a_1, a_2 \in A \)): \( f(a_1) = f(a_2) \implies a_1 = a_2 \).
(b) Give the definition of surjective (a.k.a. onto): \( f : A \to B \) is surjective when:

Answer: \( \forall b \in B \exists a \in A \ f(a) = b \).

(c) Suppose that \( f : A \to B \) and \( g : B \to A \) and (1): \( \forall a \in A \ g(f(a)) = a \).

i. Prove that \( f \) is injective.

Answer: Assume \( f(a_1) = f(a_2) \). To prove: \( a_1 = a_2 \).
Apply \( g \) to the assumed statement gives: \( g(f(a_1)) = g(f(a_2)) \).
Applying (1) to the last equation gives \( a_1 = a_2 \).

ii. Prove that \( g \) is surjective.

Answer: \( g : B \to A \) so we have to prove that if \( a \in A \) then there exists \( b \in B \) with \( g(b) = a \).
Proof: Take \( b := f(a) \) (then \( g(b) = g(f(a)) \) equals \( a \) by (1)).

iii. If \( f \) is surjective then show that \( g \) is injective.

Assume \( g(b_1) = g(b_2) \), to prove: \( b_1 = b_2 \).
Since \( f \) is surjective, there are \( a_1, a_2 \in A \) with \( b_1 = f(a_1) \) and \( b_2 = f(a_2) \). Applying \( g \) we get \( g(b_1) = g(f(a_1)) = a_1 \) (last equation used (1)). Likewise \( g(b_2) = a_2 \). But we assumed \( g(b_1) = g(b_2) \) and so \( a_1 = a_2 \). Then \( b_1 = f(a_1) = f(a_2) = b_2 \).

4. (a) Let \( x \) and \( y \) be real numbers: Consider the statement

\[
(\forall \epsilon > 0 \ x < y + \epsilon) \implies x \leq y
\]

Write down the contrapositive of this statement and simplify your answer so that you have no negation symbol in front of a quantifier.

Answer: \( x > y \implies (\exists \epsilon > 0 \ x \geq y + \epsilon) \).

(b) Can you prove the statement?

Answer: Assume \( x > y \).
Take \( \epsilon := x - y \) (then \( \epsilon > 0 \) and \( x \geq y + \epsilon \)).

5. Bonus or take-home question: Suppose that \( A, B, I \) are sets, and \( C_i \) is a set for every \( i \in I \). Suppose that \( C_i \subseteq B \) for every \( i \in I \). Show that

\[
A \setminus B \subseteq \bigcap_{i \in I} A \setminus C_i
\]

(Note: \( A \setminus B \) is the same as \( A \setminus B \))