ZFC axioms of set theory

(the axioms of Zermelo, Fraenkel, plus the axiom of Choice)

For details see Wikipedia "Zermelo-Fraenkel set theory". Note that the descriptions there are quite formal (They need to be, because the goal is to reduce the rest of math to these axioms. So to avoid circular reasoning, you have to state the axioms without using anything you know from the rest of math! That can be done but it does make the precise statements of the axioms too technical for this class).

Since the formal description may be difficult to read at this point, I typed a more informal description here, and added explanations:

A1 **Axiom of Extensionality**.
This Axiom says that two sets are the same if their elements are the same. You can think of this axiom as defining what a set is.

A2 **Axiom of Regularity**.
I have never seen a set A for which A is an element of A, but that does not imply that such sets do not exist. Do they exist? How would you construct such a set without using a circular construction? Circular definitions are forbidden in math because if you allow them, then it is very easy to run into contradictions. Now about this axiom.

This axiom looks quite technical. It implies that a set A cannot cannot be an element of A. Moreover, it also implies that A cannot be an element of an element of A. Also: A cannot be an element of an element of an element of A, etc.

A3 **Axiom schema of specification**.
If $P(x)$ is a statement (if you plug in a value for x then $P(x)$ becomes either true or false) and if A is a set, then this axiom says that the elements of A that satisfy P also form a set. We denote that set as $\{x \in A \mid P(x)\}$.

Read that notation as: The set of all $x \in A$ for which $P(x)$ is true.
The symbol | means: "for which" (some authors use : instead of |).

A4 **Axiom of Pairing** .
Says that if x, y are sets then $\{x, y\}$ is also a set.

You may wonder: why do we need this axiom? Isn't this obviously true? Well, the issue is, it doesn't matter if people feel that this is obviously true, either way, you can not actually *prove* this statement. The point of the axioms is to explicitly write down *all* statements we accept without proof, including those that some might feel are "obvious".
Think of it this way, supposed you've played chess for many years. Then you might feel that the rules of chess are obviously true. But you can't mathematically *prove* that those rules are true. So then why are the rules of chess true? Because chess organizations have decided that those are the rules they want. Likewise, in the game of math, we have settled on certain rules, and these rules are the axioms and the rules of logic.

A5 **Axiom of Union**.
If A is a set whose elements are again sets, then the union of those sets is again a set.

A6 **Axiom of replacement**.
Suppose $\phi(x, y)$ is a statement such that for every set x there is precisely one set y for which $\phi(x, y)$ is true. Denote that y as $f(x)$.

Remark: f looks a lot like a function, but we may not yet call f a function. In math you may only call f a function after specifying the domain AND the co-domain.

The axiom says that if we apply f to all elements of some set A, so take $f(x)$ for every $x \in A$, then the axiom says that all those $f(x)$ form a set. We denote that set as $\{f(x) \mid x \in A\}$.

A7 **Axiom of infinity**.
Says that there exists an infinite set. Note: It does not matter which infinite set, as long as we have at least one infinite set then the other axioms allow us to construct our favorite infinite set, which is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

You may wonder, why do we need an axiom that tells us that an infinite set exists, when we've already known about $\{0, 1, 2, 3, ...\}$ for a long time? The issue here is: How do you define the $\ldots$ in the notation $\{0, 1, 2, 3, \ldots\}$ without using circular reasoning? The only thing we can do here is to accept the unprovable statement that there exists at least one infinite set, and then work from there.

So wait, any time we need an unproven statement, we can simply call it an axiom? No, because other mathematicians won't accept additional axioms. Although the current axioms are not proven, they are supported by good evidence. They have been thoroughly put to the test for at least a century. It is outside of the scope of this course to detail the many ways in which the axioms have been investigated, but the upshot is that mathematicians are very confident that the standard axioms (called ZFC), combined with the rules of logic, do not lead to errors. Mathematicians are unlikely to accept more axioms; we do not need more axioms, and we are confident about the ones we have.

A8 **Axiom of the Power set**.
If A is a set then $P(A)$ is a set.

A9 **Axiom of Choice**. (AC)
Chapter 3 proves that this axiom is equivalent to several other statements:
1) Zorn's lemma
2) The well-ordering theorem
3) For any cardinals d, e we have $d \leq e$ or $e \leq d$.
4) For any sets A, B, if there is a surjective function from A to B then there is an injective function from B to A.

If you replaced AC by one of these four statements, then ZFC set theory stays the same. The axiom of choice, says that if A is a set whose elements are non-empty sets, then one can pick an element from each of these non-empty sets.
This sounds harmless, however, if A is an infinite set, then we have to choose one element from infinitely many sets. How do you make infinitely many choices? Even though we may not be able to do this ourselves, AC says that there does exist a function that makes all these choices. In case you think that this is fishy, that is not unreasonable, but consider this:
One can prove (using only A1-A8) that if A1-A9 is contradictory, then A1-A8 must also be contradictory. So if you want to prove that something in A1-A9 is wrong, then you also have to prove that something in A1-A8 is wrong (remember from the comments under A7 that the axioms have been put to the test in many ways).

There you have it, a full list of all statements that mathematicians accept without proof. Any statement other than A1-A9 will only be accepted by other mathematicians if it has a proof that only uses the rules of logic and the axioms A1-A9. (The use of other statements is allowed provided that they can be proved from the rules of logic and the axioms.)

Proved statements are called theorems, and mathematicians trust them even when they are counter-intuitive (like the existence of infinite sets with different cardinalities!). Conversely, other than A1-A9, mathematicians don't accept statements that don't have a proof, no matter how plausible they may sound.