Test 2, Intro Advanced Math, Oct 18 2019.

If the test is too long for the allotted time, then circle one of Ex 4 or Ex 5 and turn that into a take-home question.

1. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$
   No explanation is needed for (a)–(e). Do explain your answers for (f),(g).

   (a) $\{3, 3, 3\}$

   (b) $\mathbb{R} \setminus \mathbb{Q}$

   (c) $\mathbb{N} \times \mathbb{Q}$

   (d) $P(\mathbb{Q})$

   (e) $\mathbb{R}^{\mathbb{R}}$

   (f) $\mathbb{R}^{\mathbb{N}}$

   (g) Does there exist a bijection between $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}$?

2. Give the definitions:

   (a) A relation R on a set S is an *equivalence relation* when:

      i.

      ii.

      iii.

   (b) A function $f : A \to B$ is *onto* (*surjective*) when:

3. Suppose that $f : P(A) \to B$ is injective.
   Prove that there is no injective function $g : B \to A$.

4. Suppose that A_q is a set for every $q \in \mathbb{Q}$.
   Suppose that for every $r \in \mathbb{R}$ there is some $q \in \mathbb{Q}$ for which $r \in A_q$.
   Must there be some $q \in \mathbb{Q}$ for which A_q is uncountable?
   Why or why not?

5. Let C be a set, let A be a subset of C, and let $B = C \setminus A$. Suppose there is an injective function from B to A but not from C to A. Prove that C must be a finite set.

List of facts on cardinal numbers, shortened version.

1. $\text{card}(A) = \text{card}(B)$ means $\exists f : A \to B$ with f bijection.
2. $\text{card}(A) \le \text{card}(B)$ means $\exists f : A \to B$ with f one-to-one.
3. $\aleph_0$ is short notation for $\text{card}(\mathbb{N}^*)$.
4. c is short notation for $\text{card}(\mathbb{R})$.
5. The set A is *countably infinite* when: $\text{card}(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with f bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of A fit into one sequence $f(1), f(2), \ldots$.
6. Notation: $x < y$ is short for: $x \le y \wedge x \ne y$.
7. $\text{card}(A) < \text{card}(P(A))$.
8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $\text{card}(\mathbb{N}^*) = \aleph_0$, is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!
9. If $f : A \to B$ is onto then $\text{card}(B) \le \text{card}(A)$.
10. A is *countable* when either: A is countably infinite (defined in item 5) or A is finite.
11. A is countable when $\text{card}(A) \le \aleph_0$.
12. A subset of a countable set is again countable.
13. If $A \subseteq B$ then $\text{card}(A) \le \text{card}(B)$.
14. The ordering $\le$ on cardinal numbers is a *partial ordering*.
    In particular: whenever $d \le e$ and $e \le d$ we may conclude $d = e$.
    The proof is not easy! (Schroeder-Bernstein theorem on p 88–89).
15. The ordering $\le$ on cardinal numbers is a *total ordering*. So given any two cardinals d, e we have $d \le e$ or $d \ge e$. This means that one of these things must be true: $d < e$ or $d = e$ or $d > e$.
16. Set A is uncountable when $\text{card}(A) \not\le \aleph_0$. Using item 15 we can reformulate this by saying: A is uncountable when $\text{card}(A) > \aleph_0$.
17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).
18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.
19. If you have countably many sets, and if each of these sets is countable, then their union is also countable.

20. $\mathbb{R}$ is uncountable. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*))$.

21. If $d = \text{card}(D)$ and $e = \text{card}(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$. Likewise, $d \cdot e$ is the cardinality of $D \times E$. d^e is the cardinality of D^E where $D^E = \{\text{all functions from } E \text{ to } D\}$.

22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same!

23. There is a bijection between $P(A)$ and $\{0,1\}^A$, and hence $\text{card}(P(A)) = \text{card}(\{0,1\}^A) = \text{card}(\{0,1\})^{\text{card}(A)} = 2^{\text{card}(A)}$.

24. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*)) = \text{card}(\{0,1\}^{\mathbb{N}^*}) = 2^{\text{card}(\mathbb{N}^*)} = 2^{\aleph_0}$.

25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$

26. If you have d sets, and each of these sets has cardinality e, and if A is the union of all those sets, then $\text{card}(A) \leq de$ (if the d sets are disjoint, then you may replace the $\leq$ by $=$). Now if d or e is infinite, and both are non-zero, then we can also replace de by $\max(d,e)$, see item 22.

27. So far we have encountered these increasing cardinals:

    $$0, \ 1, \ 2, \ 3, \ \ldots \ \aleph_0, \ \ c = 2^{\aleph_0}, \ \ 2^c, \ \ 2^{2^c}, \ \ldots$$

    and we can wonder if there are any cardinals in between. Specifically, the *continuum hypothesis* asks if there is a cardinal d with $\aleph_0 < d < c$.

    From the axioms of set theory (= the only statements mathematicians accept without a proof) it is impossible to prove or disprove this.

Writing Proofs.

1. **Direct proof for** $p \implies q$**.**
   Assume: p. To prove: q.

2. **Proving** $p \implies q$ **by contrapositive**.
   Assume: $\neg q$. To prove: $\neg p$.

3. **Proving** S **by contradiction**.
   Assume: $\neg S$. To prove: a contradiction.

4. **Proving** $p \implies q$ **by contradiction**.
   Assume: p and $\neg q$. To prove: a contradiction.

5. **Direct proof for a** $\forall_{x \in A} P(x)$ **statement**.
   To ensure you prove $P(x)$ for *all* (rather than for *some*) x in A, do this:
   Start your proof with: Let $x \in A$. To prove: $P(x)$.

6. **Direct proof for** $\exists_{x \in A} P(x)$ **statement**.
   Take $x :=$ [write down an expression that is in A, and satisfies $P(x)$].

7. **Proving** $\forall_{x \in A} P(x)$ **by contradiction.**
   Assume: $x \in A$ and $\neg P(x)$. To prove: a contradiction.

8. **Proving** $\exists_{x \in A} P(x)$ **by contradiction.**
   Assume: $\neg P(x)$ for every $x \in A$. To prove: a contradiction.

9. **Proving** S **by cases**.
   Suppose for example a statement p can help to prove S. Write two proofs:
   Case 1: Assume p. To prove: S.
   Case 2: Assume $\neg p$. To prove S.

10. **Proving** $p \wedge q$
    Write two separate proofs: To prove: p. To prove: q.

11. **Proving** $p \iff q$
    Write two proofs. To prove: $p \implies q$ To prove: $q \implies p$.

12. **Proving** $p \vee q$
    Method (1): Assume $\neg p$. To prove: q.
    Method (2): Assume $\neg q$. To prove: p.
    Method (3): Assume $\neg p$ and $\neg q$. To prove: a contradiction.

13. **Using** $p \vee q$ **to prove another statement** r.
    Write two proofs:
    Assume p. To prove r.
    Assume q. To prove r.

14. **How to use a for-all statement** $\forall_{x \in A} P(x)$.
    You need to produce an element of A, then use P for that element.

15. If you want to **use an exists statement** like $\exists_{x \in A} P(x)$ to prove another statement, then you *may not choose* x. All you know is $x \in A$ and $P(x)$.