More sample questions for Intro Advanced Math.

1. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$

   For (a)–(h) you do not need to show your work, but for (i),(j) you need to justify your answer by showing all steps.

   (a) $\mathbb{N}$
   (b) $\emptyset \times \mathbb{R}$
   (c) $\mathbb{Q}$
   (d) $\mathbb{R} \times P(\mathbb{Q})$
   (e) $\mathbb{Q} - \mathbb{Z}$
   (f) $P(\mathbb{N})$
   (g) $P(\mathbb{R})$
   (h) $\{2, 2\}$
   (i) $\mathbb{R}^{\mathbb{N}}$
   (j) $\mathbb{R}^{\mathbb{R}}$

2. Based on the answer in your previous question, does there exist: (it suffices to write yes/no):

   (a) an injective function from $\mathbb{R}^{\mathbb{R}}$ to $P(\mathbb{R})$?
   (b) an injective function from $\mathbb{Q}$ to $\mathbb{N}$?
   (c) an injective function from $P(\mathbb{N})$ to $\mathbb{N}$?

3. Prove, using only the definition, that the intervals $(0, 1)$ and $(0, 2)$ have the same cardinality.

4. Let A, B be sets and let $C = A \bigcup B$. Suppose that $A \bigcap B = \emptyset$ and:

   - there is **no** bijection from A to C
   - there is **no** bijection from B to C

   Prove that A and B are finite sets.

5. Let A be any set. Prove that there is no bijection from $\mathbb{N}$ to $P(A)$.

6. TURN IN:
We know that if d, e are natural numbers then $d \cdot e = e \cdot d$. But do you remember how to prove that? Lets prove this not only for natural numbers, but for all cardinal numbers! I will type the first line in the proof, and you finish it:

   Proof: Let D, E be sets for which $d = o(D)$ and $e = o(E)$.

   (a) Give the definition of $D \times E$.
   (If you are not 100% sure then read section 1.5).
   (b) Give a bijection from $D \times E$ to $E \times D$.

   (Note: Don't write a lot of text. The thing you have to write down is a recipe that takes as input: an element of $D \times E$, and gives as output: an element of $E \times D$.)
   (Also: don't use the letter d to denote elements of D because we already used d for something else. You can for example use the letters x resp. y to denote an element of D resp. E).
   (c) Why does this bijection prove $d \cdot e = e \cdot d$?

   (Study "List of facts on cardinal numbers" and look for $d \cdot e$).

7. TURN IN:
Find all sets A for which the following is true:
Every element of A is equal to 1.

8. TURN IN:
Item 21 says that if d, e are cardinals, and if at least one of them is infinite, then $d + e = \max(d, e)$. It is quite hard to prove this in general. Lets prove it in a special case, when $d = e = \aleph_0$, as follows:
Let $\mathbb{N}^* = \{1, 2, 3, 4, \ldots\}$, $E = \{2, 4, 6, 8, \ldots\}$, $D = \{1, 3, 5, 7, \ldots\}$.
So $E = \{\text{all even positive integers}\}$, and $D = \{\text{all odd positive integers}\}$.

   (a) Give a bijection $f : \mathbb{N}^* \to E$ (write down: $f(n) = \ldots$)
   (b) Give a bijection $g : \mathbb{N}^* \to D$.
   (c) Explain why parts (a),(b) prove that $\aleph_0 + \aleph_0 = \aleph_0$.

   (Study "List of facts on cardinal numbers" and look for $d + e$).

9. Study past quizzes, tests, handouts, HW, and class notes.
Bring questions!

List of facts on cardinal numbers, shortened version.

Note: During the actual test, basic definitions that everyone must know (such as items 1–7) may be deleted!

1. $o(A) = o(B)$ means $\exists f : A \to B$ with f bijection.

2. $o(A) \leq o(B)$ means $\exists f : A \to B$ with f one-to-one.

3. $\aleph_0$ is short notation for $o(\mathbb{N}^*)$.

4. c is short notation for $o(\mathbb{R})$.

5. The set A is *countably infinite* when: $o(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with f bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of A fit into one sequence $f(1), f(2), \ldots$.

6. Notation: $x < y$ is short for: $x \leq y \wedge x \neq y$.

7. $o(A) < o(P(A))$.

8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $o(\mathbb{N}^*) = \aleph_0$, is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!

9. If $f : A \to B$ is onto then $o(B) \leq o(A)$.

10. A is *countable* when either: A is countably infinite (defined in item 5) or A is finite.

11. A is countable when $o(A) \leq \aleph_0$.

12. A subset of a countable set is again countable.

13. If $A \subseteq B$ then $o(A) \leq o(B)$.

14. The ordering $\leq$ on cardinal numbers is a *partial ordering*.
    In particular: whenever $d \leq e$ and $e \leq d$ we may conclude $d = e$.
    You might remember that the proof was not easy!

15. The ordering $\leq$ on cardinal numbers is a *total ordering*. So given any two cardinals d, e we have $d \leq e$ or $d \geq e$. This means that one of these things must be true: $d < e$ or $d = e$ or $d > e$.

16. Set A is uncountable when $o(A) \not\leq \aleph_0$. Using item 15 we can reformulate this by saying: A is uncountable when $o(A) > \aleph_0$.

17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).

18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

19. If you have countably many sets, and if each of these sets is countable, then their union is also countable.

20. $\mathbb{R}$ is uncountable. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*))$.

21. If $d = o(D)$ and $e = o(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$. Likewise, $d \cdot e$ is the cardinality of $D \times E$. d^e is the cardinality of D^E where $D^E = \{\text{all functions from } E \text{ to } D\}$.

22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same!

23. There is a bijection between $P(A)$ and $\{0,1\}^A$, and hence $o(P(A)) = o(\{0,1\}^A) = o(\{0,1\})^{o(A)} = 2^{o(A)}$.

24. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*)) = o(\{0,1\}^{\mathbb{N}^*}) = 2^{o(\mathbb{N}^*)} = 2^{\aleph_0}$.

25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$

26. If you have d sets, and each of these sets has cardinality e, and if A is the union of all those sets, then $o(A) \leq de$ (if the d sets are disjoint, then you may replace the $\leq$ by $=$). Now if d or e is infinite, and both are non-zero, then we can also replace de by max(d,e), see item 22.

27. So far we have encountered these increasing cardinals:

    $$0,\ 1,\ 2,\ 3,\ \ldots\ \aleph_0,\ \ c = 2^{\aleph_0},\ \ 2^c,\ \ 2^{2^c},\ \ldots$$

    and we can wonder if there are any cardinals in between. Specifically, the *continuum hypothesis* asks if there is a cardinal d with $\aleph_0 < d < c$.

    From the axioms of set theory (= the only statements mathematicians accept without a proof) it is impossible to prove or disprove this.