Intro Advanced Math, definitions for sets

Make sure to know all these definitions, we will need them all the time.

1. Sets are the only objects in mathematics that are not defined in terms of previously-defined objects. Instead, they are described indirectly by stating their properties (more details on that later in this course).

   Given any set A and any object x, the phrase "$x \in A$" is a *statement* (which means: it is either True or False).
   It is pronounced as: x is an element of A.
   The negation of this statement is $x \notin A$ (x is not an element of A).

2. **Set equality** is defined by the following rule:

   $$A = B \quad \text{means} \quad x \in A \Longleftrightarrow x \in B. \tag{1}$$

   Strictly speaking it is $\forall_x \, (x \in A \Longleftrightarrow x \in B)$ but this $\forall_x$ is often omitted.

3. **Subset:**

   $$A \subseteq B \quad \text{means} \quad x \in A \Longrightarrow x \in B. \tag{2}$$

   Strictly speaking it is $\forall_x \, (x \in A \Longrightarrow x \in B)$ but we again omitted $\forall_x$.
   The notation $\subset$ means the same as $\subseteq$.

4. Since $p \Longleftrightarrow q$ is logically equivalent to $(p \Longrightarrow q) \wedge (q \Longrightarrow p)$ it follows from (1)+(2) that

   $$A = B \quad \text{is equivalent to} \quad A \subseteq B \;\wedge\; B \subseteq A \tag{3}$$

   So to prove sets are equal you can use either (1) or (3).

5. The **empty set** is denoted as $\emptyset$ and is defined by $\forall_x \; x \notin \emptyset$. Then (1) says

   $$A = \emptyset \quad \text{means} \quad \forall_x \; x \notin A. \tag{4}$$

   One way to prove $A = \emptyset$ starts with:
   Let x be any object. To prove $x \notin A$.

   A proof by contradiction (WP#3) for $A = \emptyset$ starts with:
   Assume $x \in A$. To prove: a contradiction.

6. Example: Suppose $A \subseteq \emptyset$. Then prove: $A = \emptyset$.
   Proof: Let $x \in A$. To prove: a contradiction.
   From $x \in A$ and $A \subseteq \emptyset$ it follows[1] that $x \in \emptyset$ which is always false.

[1] To see this you need to know (2)!

7. Example: Let A be a set. Then prove $\emptyset \subseteq A$.
To Prove: $x \in \emptyset \Longrightarrow x \in A$.
Proof: $x \in \emptyset$ is always false, and "false$\Longrightarrow$anything" is always true.
[We used (2) in the T.P. statement, and (4) + truth tables for the proof.]
[Text in square brackets [ ] are additional comments that you don't need to write in your proofs.]

8. Definition of the **power set** (the set of all subsets): $P(A) = \{S \mid S \subseteq A\}$ or equivalently:
$$S \subseteq A \iff S \in P(A) \tag{5}$$

9. Example: $\emptyset \subseteq A$ and hence $\emptyset \in P(A)$. Likewise $A \subseteq A$ and so $A \in P(A)$. In particular $P(A)$ is never empty.
[$P(\emptyset) = \{\emptyset\} \neq \emptyset$. "A bag with an empty bag in it" $\neq$ "an empty bag".]

10. Example: Suppose $P(A) \subseteq P(B)$. Prove $A \subseteq B$.
Proof: $A \subseteq A$ so $A \in P(A)$ but $P(A) \subseteq P(B)$ so $A \in P(B)$ hence $A \subseteq B$.
[First we used (5), then used (2) for $P(A) \subseteq P(B)$, then used (5) again. To write proofs, you really need to know the definitions (1),(2),...!]

11. (a, b) is an **ordered pair**. For example $\{1, 2\} = \{2, 1\}$ but $(1, 2) \neq (2, 1)$.

12. $A \times B = \{(a, b) \ : \ a \in A, b \in B\}$ = the set of all pairs (a, b) for all $a \in A$ and all $b \in B$. So
$$(a, b) \in A \times B \quad \text{is the same as} \quad a \in A \wedge b \in B \tag{6}$$

13. Exercise: Show that $A \times \emptyset = \emptyset$. Proof:
An element of $A \times \emptyset$ is a pair: (an element of A, an element of $\emptyset$). There are no such pairs because $\emptyset$ has no elements.

14. Definition of $\bigcap$ $\bigcup$ and $\setminus$ (instead of $A \setminus B$ you may also write $A - B$)
$$x \in A \bigcap B \iff (x \in A \ \wedge \ x \in B) \tag{7}$$
$$x \in A \bigcup B \iff (x \in A \ \vee \ x \in B) \tag{8}$$
$$x \in \bigcap_{i \in I} A_i \iff \forall_{i \in I} \ x \in A_i \tag{9}$$
$$x \in \bigcup_{i \in I} A_i \iff \exists_{i \in I} \ x \in A_i \tag{10}$$
$$x \in A \setminus B \iff (x \in A \ \wedge \ x \notin B) \tag{11}$$

15. Example: Prove $(A \setminus B) \bigcap B = \emptyset$.
[Read item 5] Proof: Assume $x \in (A \setminus B) \bigcap B$. To prove: a contradiction. By (7) we get $x \in A \setminus B$ and $x \in B$. Then by (11) we get $x \in A$ and $x \notin B$ which contradicts $x \in B$.

16. Make sure to know truth tables, and how to negate any statement.

17. If we have selected a universal set U, it means that for any element x under consideration we automatically assume $x \in U$, and for any set A under consideration, we automatically assume $A \subseteq U$. Then the notation A' denotes $U \setminus A$. Under these assumptions, we may use

$$x \in A' \iff x \notin A \tag{12}$$

More examples:

- **Example:** Prove that $(A' \bigcup B')' = A \bigcap B$.
  This means [see (1)] to prove: $x \in$ left-hand-side $\iff x \in$ right-hand-side.

  Proof: $x \in$ left-hand-side is equivalent (see (12)) to $\neg(x \in A' \bigcup B')$ which is equivalent (see (8)) to $\neg(x \in A' \vee x \in B')$ which is equivalent to $\neg(x \notin A \vee x \notin B)$ which is equivalent by De Morgan's law to $x \in A \wedge x \in B$, which is equivalent to $x \in$ right-hand-side.

- **Example:** Let $I = (0, \infty) =$ set of positive real numbers. Let $A_i = (-i, i)$ (not an ordered pair, but the set of all real numbers strictly between $-i$ and i, which unfortunately has the same notation). Prove that $\bigcap_{i \in I} A_i = \{0\}$.

  To prove: (1): $\{0\} \subseteq \bigcap_{i \in I} A_i$ and (2): $\bigcap_{i \in I} A_i \subseteq \{0\}$.
  (1): same as $0 \in \bigcap_{i \in I} A_i$, same as $\forall_{i \in I}\ 0 \in A_i$ [see (9)] which is true.
  For (2), we have to show $x \in \bigcap_{i \in I} A_i \Longrightarrow x \in \{0\}$.
  Proof by contrapositive [WP#2]: Suppose $x \notin \{0\}$ in other words $x \neq 0$.
  To Prove: $\neg(x \in \bigcap_{i \in I} A_i)$, which [see (9)] is the same as $\neg(\forall_{i \in I}\ x \in A_i)$ which is equivalent to $\exists_{i \in I}\ x \notin A_i$ [To prove that $\exists$ we will follow WP#6]:
  Proof: Take $i = |x|/2$. [also OK: $|x|$ or $|x|/1000$ etc.]

Answers to the HW questions that were due Jan 14:

1. Let x be an odd integer. Prove that $4|x^2 - 1$.

   Proof: From the definition of odd integer, we can write $x = 2k + 1$ for some $k \in \mathbb{Z}$. Then $x^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$. So $x^2 - 1 = 4 \cdot$ (an integer) in other words $4|x^2 - 1$.

2. For any set A, prove that $\emptyset \subseteq A$.

   Proof: From the definition of $\subseteq$ (equation (2) on page 1) we see that we have to prove (for any x) that $x \in \emptyset \Longrightarrow x \in A$. The definition of $\emptyset$ on page 1 says that $x \in \emptyset$ is always false. So our to-prove statement is of the form false $\Longrightarrow$ something which we know from the truth-tables to be true.

3. Suppose that $B \subseteq A$ for any set A. Prove that $B = \emptyset$.

   Proof: Choose $A = \emptyset$ [read WP#14] so $B \subseteq \emptyset$. The definitions on page 1 then say (for any x) that $x \in B \Longrightarrow x \in \emptyset$ and that $x \in \emptyset$ is false. So $x \in B \Longrightarrow$ false but that is logically equivalent to $\neg(x \in B)$ (i.e. $x \notin B$). But $x \notin B$ (still for any x) means $B = \emptyset$.