# Intro Advanced Math, test 1. **Your name:**

1. Let $A, B$ be sets and suppose that $A - B = B$. Then show that $A$ and $B$ are both empty.

   Answer: To prove $B = \emptyset$ (i.e. $\forall_x\ x \notin B$) we will show that $x \in B$ leads to a contradiction.
   Assume $x \in B$. Then $x \in A - B$ since $A - B = B$, so $x \in A$ and $x \notin B$, contradicting the assumption.
   Now that we know $B = \emptyset$ we find $A = A - \emptyset = A - B = B = \emptyset$.

2. (a) Let $f$ be a function from $A$ to $B$. Write down the *contrapositive* of this statement:

   $p$: $\quad f(x) = f(y) \Longrightarrow x = y$

   Answer: $x \neq y \Longrightarrow f(x) \neq f(y)$.

   (b) Write down the *converse* of statement $p$.

   Answer: $x = y \Longrightarrow f(x) = f(y)$.

   (c) Is there a statement among your answers for (a),(b) that is true for every function?

   Yes, (b) says that $f$ is well-defined. That is true for every function.

   (d) Now compute the *negation* of this statement:

   $q$: $\quad$ There exists $b \in B$ such that $b \neq f(a)$ for every $a \in A$.

   Answer: $\neg q$ says $\forall_{b \in B} \exists_{a \in A}\ b = f(a)$.

   (e) Can you express your answer for $\neg q$ in terms of one of the phrases/definitions you memorized?

   $\neg q$ says that $f$ is onto.

   (f) Let $L$ be a chain, let $S$ be a subset of $L$, and consider this statement:

   $r$: $\quad$ For every $x$ in $S$ there exists $y$ in $S$ with $y > x$.

   Compute the *negation* of $r$ (Recall that in a chain, the negation of $y > x$ is simply $y \leq x$).

   Answer: $\neg r$ says $\exists_{x \in S} \forall_{y \in S}\ y \leq x$.

   (g) Can you express your answer for $\neg r$ in terms of one of the phrases we have learned?

   Answer: $\neg r$ says: $\exists_{x \in S}$ such that $x$ is an upper bound for $S$.
   An upper bound for $S$ that happens to be in $S$ is called a top element.
   So $\neg r$ says that $S$ has a top element.

3. (a) Give the definition of injective (a.k.a. one to one): $f : A \to B$ is injective when:

   Answer: (for all $a_1, a_2 \in A$): $\ f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$.

(b) Give the definition of surjective (a.k.a. onto): $f : A \to B$ is surjective when:

Answer: $\forall_{b \in B} \exists_{a \in A} \; f(a) = b$.

(c) Suppose that $f : A \to B$ and $g : B \to A$ and (1): $\forall_{a \in A} \; g(f(a)) = a$.

i. Prove that $f$ is injective.

Answer: Assume $f(a_1) = f(a_2)$. To prove: $a_1 = a_2$.
Apply $g$ to the assumed statement gives: $g(f(a_1)) = g(f(a_2))$.
Applying (1) to the last equation gives $a_1 = a_2$.

ii. Prove that $g$ is surjective.

Answer: $g : B \to A$ so we have to prove that if $a \in A$ then there exists $b \in B$ with $g(b) = a$.
Proof: Take $b := f(a)$ (then $g(b) = g(f(a))$ equals $a$ by (1)).

iii. If $f$ is surjective then show that $g$ is injective.

Assume $g(b_1) = g(b_2)$, to prove: $b_1 = b_2$.
Since $f$ is surjective, there are $a_1, a_2 \in A$ with $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Applying $g$ we get $g(b_1) = g(f(a_1)) = a_1$ (last equation used (1)). Likewise $g(b_2) = a_2$. But we assumed $g(b_1) = g(b_2)$ and so $a_1 = a_2$. Then $b_1 = f(a_1) = f(a_2) = b_2$.

4. (a) Let $x$ and $y$ be real numbers: Consider the statement

$$(\forall_{\epsilon > 0} \;\; x < y + \epsilon) \implies x \leq y$$

Write down the *contrapositive* of this statement and simplify your answer so that you have no negation symbol in front of a quantifier.

Answer: $x > y \implies (\exists_{\epsilon > 0} \;\; x \geq y + \epsilon)$.

(b) Can you prove the statement?

Answer: Assume $x > y$.
Take $\epsilon := x - y$ (then $\epsilon > 0$ and $x \geq y + \epsilon$).

5. Bonus or take-home question: Suppose that $A, B, I$ are sets, and $C_i$ is a set for every $i \in I$. Suppose that $C_i \subseteq B$ for every $i \in I$. Show that

$$A \setminus B \;\subseteq\; \bigcap_{i \in I} A \setminus C_i$$

(Note: $A \setminus B$ is the same as $A - B$)

## A few formulas

$$A = B \quad \text{means} \quad x \in A \Longleftrightarrow x \in B. \tag{1}$$

$$A \subseteq B \quad \text{means} \quad x \in A \Longrightarrow x \in B. \tag{2}$$

$$A = \emptyset \quad \text{means} \quad \forall_x \; x \notin A. \tag{3}$$

$$x \in \bigcap_{i \in I} A_i \quad \text{means} \quad \forall_{i \in I} \; x \in A_i \tag{4}$$

$$x \in \bigcup_{i \in I} A_i \quad \text{means} \quad \exists_{i \in I} \; x \in A_i \tag{5}$$

# Writing Proofs.

1. **Direct proof for $p \Longrightarrow q$.**
   Assume: $p$. To prove: $q$.

2. **Proving $p \Longrightarrow q$ by contrapositive**.
   Assume: $\neg q$. To prove: $\neg p$.

3. **Proving $S$ by contradiction**.
   Assume: $\neg S$. To prove: a contradiction.

4. **Proving $p \Longrightarrow q$ by contradiction**.
   Assume: $p$ and $\neg q$. To prove: a contradiction.

5. **Direct proof for a $\forall_{x \in A} P(x)$ statement**.
   To ensure you prove $P(x)$ for *all* (rather than for *some*) $x$ in $A$, do this:
   **Start your proof with:** Let $x \in A$. To prove: $P(x)$.

6. **Direct proof for $\exists_{x \in A} P(x)$ statement**.
   Take $x :=$ [write down an expression that is in $A$, and satisfies $P(x)$].

7. **Proving $\forall_{x \in A} P(x)$ by contradiction.**
   Assume: $x \in A$ and $\neg P(x)$. To prove: a contradiction.

8. **Proving $\exists_{x \in A} P(x)$ by contradiction.**
   Assume: $\neg P(x)$ for every $x \in A$. To prove: a contradiction.

9. **Proving $S$ by cases**.
   Suppose for example a statement $p$ can help to prove $S$. Write two proofs:
   Case 1: Assume $p$. To prove: $S$.
   Case 2: Assume $\neg p$. To prove $S$.

10. **Proving $p \wedge q$**
    Write two separate proofs: To prove: $p$. To prove: $q$.

11. **Proving $p \Longleftrightarrow q$**
    Write two proofs. To prove: $p \Longrightarrow q$ To prove: $q \Longrightarrow p$.

12. **Proving $p \vee q$**
    Method (1): Assume $\neg p$. To prove: $q$.
    Method (2): Assume $\neg q$. To prove: $p$.
    Method (3): Assume $\neg p$ and $\neg q$. To prove: a contradiction.

13. **Using $p \vee q$ to prove another statement** $r$.
    Write two proofs:
    Assume $p$. To prove $r$.
    Assume $q$. To prove $r$.

14. **How to use a for-all statement** $\forall_{x \in A} P(x)$.
    You need to produce an element of $A$, then use $P$ for that element.

15. If you want to **use an exists statement** like $\exists_{x \in A} P(x)$ to prove another statement, then you *may not choose* $x$. All you know is $x \in A$ and $P(x)$.