Intro Advanced Math., test 2, October 21, 2015

1. Simplify the following cardinal numbers to the point where they look like $0, 1, 2, \ldots$, or $\aleph_0$, or c, or 2^c, or $2^{(2^c)}$, etc.

$\aleph_0 + \aleph_0$	$\aleph_0 + c + 2^{\aleph_0}$
$o(\{3, 3, 3\})$	$(2^c)^{c^2}$
$o(P(\emptyset))$	$o(P(\mathbb{Q}) - P(\mathbb{N}))$
$o(P(\mathbb{R})^{\mathbb{R}})$	$o(\emptyset^{\mathbb{R}} \times \mathbb{R})$
$o(\mathbb{R} \times \mathbb{N})$	$o(\mathbb{R}^{\mathbb{N}})$
$7^{\aleph_0}$	$o(P(\mathbb{N}))$

2. Let A, B, C be sets.

   (a) Write down the definition of $o(A) \leq o(B)$.

   (b) We know that

   $$\text{if} \quad o(A) \leq o(B) \quad \text{and} \quad o(B) \leq o(C) \quad \text{then} \quad o(A) \leq o(C).$$

   Write down a proof for this rule using only the definition of $\leq$.

3. (a) Give the definition of cardinal addition: if $D \bigcap E = \emptyset$ and $d = o(D)$ and $e = o(E)$ then $d + e$ is defined as:

   (b) Let A be an infinite set, let B be a subset and let $C = A - B$.
   Suppose that there is a bijection from B to C.
   Prove that there is a bijection from A to C.

   (c) Give the definition of cardinal multiplication.

   (d) Let A, B, C as in part (b), and assume that B and C are not empty.
   Prove that there is a bijection from A to $B \times C$.

4. Let $\mathbb{N}^* = \{1, 2, 3, 4, \ldots\}$, $E = \{2, 4, 6, 8, \ldots\}$, $D = \{1, 3, 5, 7, \ldots\}$.
   So $E = \{\text{all even positive integers}\}$, and $D = \{\text{all odd positive integers}\}$.

   (a) Give a bijection $f : \mathbb{N}^* \to E$ (write down: $f(n) = \ldots$)

   (b) Give a bijection $g : \mathbb{N}^* \to D$.

   (c) Explain why parts (a),(b), and Ex.3(a), show: $\aleph_0 + \aleph_0 = \aleph_0$.

5. Take home: Suppose that for each $n \in \mathbb{N}^*$ you are given a subset $A_n \subseteq \mathbb{R}$. Suppose that $\mathbb{R} = A_1 \bigcup A_2 \bigcup A_3 \bigcup \cdots$. Show that at least one of those sets A_n must be uncountable.

6. Take home: Let $A = \{S \subset \mathbb{R} \mid S \text{ countable}\}$. So A is the set of all *countable* subsets of $\mathbb{R}$, in other words:
   $S \in A$ if and only if S is countable and $S \subset \mathbb{R}$.

   Prove that $o(A) = c$.
   Hint: Find an onto function $F : \mathbb{R}^{\mathbb{N}^*} \to B$ where $B = A - \{\emptyset\}$.