# Handout WP: Writing Proofs.

1. **Direct proof of an if-then statement**.
   Many theorems have the form "if $p$ then $q$" where $p, q$ are statements.
   (Note: $p, q$ are often composed of other statements with and's and or's).
   A direct proof of $p \Longrightarrow q$ works like this:

   Assume: $p$.
   To prove: $q$.

2. **Proof by contrapositive**.
   The statement $p \Longrightarrow q$ is *logically equivalent to* (means: same truth-table) the statement $\neg q \Longrightarrow \neg p$. This means we can prove $p \Longrightarrow q$ like this:

   Assume: $\neg q$.
   To prove: $\neg p$.

   **Remark:** For any statement $p$ you need to be able to compute $\neg p$ (see the second handout "Quantifiers and Negation" for rules and examples on computing $\neg p$ for any statement $p$).

3. **Proof by contradiction**.
   One way to prove a statement $S$ is as follows:

   Assume: $\neg S$.
   To prove: a contradiction.

   Proving a contradiction means proving something that is obviously wrong (e.g. $x \neq x$, or the negation of a given/assumed/proved statement).

4. **Proving an if-then statement by contradiction**.
   If $S$ is the statement $p \Longrightarrow q$ then $\neg S$ is logically equivalent to $p \wedge \neg q$.
   So a proof-by-contradiction for $p \Longrightarrow q$ works like this:

   Assume: $p$ and $\neg q$.
   To prove: a contradiction.

5. **Direct proof for a for-all statement**.
   If $P(x)$ is a statement involving $x$, to prove a statement like this $\forall_{x \in A} P(x)$ you write the following:

   Let $x \in A$.
   To prove: $P(x)$.

   **Explanation:** When you write "Let $x \in A$" then you are telling the reader that $x$ is an element of $A$, but you are not specifying *which* element of $A$. That means that $x$ could be any element of $A$. Once you proved $P(x)$ for such $x$ then $P(x)$ must be true for any element of $A$.

   **What not to do:** Suppose $f : S \to T$ and you want to prove $\forall_{t \in T} P(t)$.
   You should start with: "Let $t \in T$". But what if you this do instead:
   Some steps ... take $t = f(s)$ ... some steps ... hence $P(t)$.
   Then you only proved $P(t)$ for some, but not all, elements of $T$.

6. **Direct proof for an exists statement**.
   If $P(x)$ is a statement involving $x$, to prove a statement like this $\exists_{x \in A} P(x)$ you write the following:

   Take $x :=$ [write down an expression].

   If it is obvious that the expression you wrote down meets the requirements $x \in A$ and $P(x)$ then the proof is now complete; you have a 1-line proof! (only non-obvious requirements need to be checked). If you wrote a lot of text but not this 1 line, then your proof is still not complete.

7. **Proving a for-all statement by contradiction.**
   If $S$ is the statement $\forall_{x \in A} P(x)$ then $\neg S$ is the statement $\exists_{x \in A} \neg P(x)$. So if we follow item 3 (i.e. prove $S$ by assuming $\neg S$ and then proving a contradiction) then the proof would start like this:

   Assume: $x$ is an element of $A$ and $\neg P(x)$.
   To prove: a contradiction.

8. **Proving an exists-statement by contradiction.**
   If $S$ is the statement $\exists_{x \in A} P(x)$ then $\neg S$ is the statement $\forall_{x \in A} \neg P(x)$.

   Assume: $\neg P(x)$ for every $x \in A$.
   To prove: a contradiction.

9. **Proof by cases**.
   Suppose we need to prove some statement $P(x)$ where $x$ can only have a few possible values. In that case, we can write a separate proof for each possible value.

   **Example 1:** Suppose we have to prove $P(x)$ but we know that $x$ can only be $u$ or $v$ or $w$. Then we write three proofs:
   Case 1: $x = u$. Write a proof for $P(u)$.
   Case 2: $x = v$. Write a proof for $P(v)$.
   Case 3: $x = w$. Write a proof for $P(w)$.

   **Example 2:** Suppose we have to prove a statement $p$ but there is some other statement $q$ such that we can easily find a proof for $q \implies p$. Then we can do the following, we split the proof in two cases:

   Case 1: Assume $q$ is true and prove $p$ under that assumption.
   Case 2: Assume $q$ is false now prove $p$ under that assumption.

   These cases combined provide a complete proof for $p$.

10. **Proving an and statement.**
    To prove $p \wedge q$ write two separate proofs:

    To prove: $p$

    To prove: $q$

11. **Proving an iff statement** (iff = "if and only if").
The statement $p \Longleftrightarrow q$ is logically equivalent to $(p \Longrightarrow q) \wedge (q \Longrightarrow p)$.
So to prove $p \Longleftrightarrow q$ you have to write two proofs:

    To prove: $p \Longrightarrow q$

    To prove: $q \Longrightarrow p$

    Proofs of "if and only if" statements in math books typically look like:

    Assume $p$ ... some math ... hence $q$. For the converse ...

    Recall that $q \Longrightarrow p$ is called the **converse** of $p \Longrightarrow q$. Math books assume you know that "For the converse" means "To prove: $q \Longrightarrow p$".

12. **Proving an or statement.**
The statement $p \vee q$ means that at least one of $p$ or $q$ is true. But which one? That question makes it tricky to give a direct proof of $p \vee q$. But you can always replace a statement by a logically equivalent statement. We have several options:
(1) $p \vee q$ is logically equivalent to $\neg p \Longrightarrow q$.
(2) $p \vee q$ is logically equivalent to $\neg q \Longrightarrow p$.
(3) $\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$.

    That gives us several ways to prove $p \vee q$.

    Method (1): Assume $\neg p$. To prove: $q$.

    Method (2): Assume $\neg q$. To prove: $p$.

    Method (3): Assume $\neg p$ and $\neg q$. To prove: a contradiction.

    Which method is best? That depends on what you already know. Suppose for instance you see a theorem in the book of the form $\neg q \Longrightarrow r$. In that case you want to try method (2) (assume $\neg q$) because then you can use the theorem to conclude $r$. Hopefully that brings you closer to goal $p$.

13. **Using an or statement**.
Suppose you want to prove $r$, and you are given $p \vee q$ which means that at least one of $p$ or $q$ is true. But which one? So we write two proofs:

    Assume $p$. To prove $r$.

    Assume $q$. To prove $r$.

    The two proofs combined show that $p \vee q$ implies $r$.

    This is the same as "Proof by cases" from item 9; given $p \vee q$ you distinguish two cases: Case 1: assume $p$, to prove $r$. Case 2: assume $q$, to prove $r$.

14. If you want to **use a for-all statement** like $\forall_{x \in A} P(x)$ to prove another statement, often the best strategy is to make a clever choice for one particular element of $A$, and then use the fact that $P$ is true for that element.

15. If you want to **use an exists statement** like $\exists_{x \in A} P(x)$ to prove another statement, then you *may not choose* $x$. All you know is $x \in A$ and $P(x)$.