List of facts for Chapter 4.

1. A **metric space** M is set with a distance function with the following properties (for all $a, b, c \in M$): $D(a,a) = 0$, $D(a,b) > 0$ whenever $a \neq b$, $D(b,a) = D(a,b)$, and the triangle inequality: $D(a,c) \leq D(a,b) + D(b,c)$.

2. $S_r(x)$ is the **open ball** with radius r and center x.
$S_r(x) = \{p \in M | D(x,p) < r\}$. So this is the set of all points you can reach if you start from x and then travel a distance that is *less than* r.

3. We say that p and x are r**-close** when $D(p,x) < r$.
So $S_r(x)$ is the set of all points that are r-close to x.

4. Any set that contains $S_r(x)$ for some $r > 0$ is called a **neighborhood** of x. So a set U is a neighborhood of x when there exists some positive r such that all points that are r-close to x are in the set U.

5. Let U be a subset of M. The following statements are **equivalent** (they are either all true, or all false).
   (a) $\exists_{r>0}\ S_r(x) \subseteq U$
   (b) U is a neighborhood of x
   (c) U contains a neighborhood of x.

6. A set $U \subseteq M$ is **open** when property 5(a) is true for every x in U.
(Note: if 5(a) is true then 5(b) and 5(c) are true as well.)

7. Note: a neighborhood of x is **not the same** as an open set, because if we want to check that U is an open set then we need to check property 5(a) for *every* element of U. Whereas to check if U is a neighborhood of x, we only have to check property 5(a) for one element (namely x).

8. The empty set $\emptyset$ is open. To check that a set is open, we have to check item 6, which means checking 5(a) for every x in our set. If $U = \emptyset$ then the number of x's for which we have to check 5(a) is zero: The condition in item 6 is **vacuously true**.
Even though the empty set is open, it is not a neighborhood of any point $x \in M$ because to be a neighborhood of x, you have to contain $S_r(x)$ for some $r > 0$, and that is not empty because $x \in S_r(x)$.

   For any metric space M, the set M itself is always open (even if M does not "look like an open set"!). This is because the $S_r(x)$ in item 5(a) is defined in item 2 in such a way that $S_r(x)$ is always a subset of M, so item 5(a) always holds if $U = M$.

9. An open neighborhood is (these conditions are equivalent):
   (a) A neighborhood of x that happens to be an open set.
   (b) An open set that happens to contain x.

10. **Any** union of open sets is always open (even if you take a union of infinitely many sets).

11. The intersection of **finitely many** open sets is again open.

12. x is an isolated point when:

    (a) $\{x\}$ is open

    (b) There is a neighborhood of x that contains just x and no other elements.

    (c) $\exists_{r>0}\, S_r(x) = \{x\}$
    This means that there exists some positive distance r such that if you travel in your metric space M, starting at x, traveling a distance that is less than r, then the only point in M you can reach is the point x itself. So travelers that can travel only a very small distance can, if they are at the point x, only reach the point x and no other points. From the viewpoint of those travelers you can see that it is reasonable terminology to say that the point x is isolated.

    Example 1: if M is a finite set, then every point $x \in M$ is isolated. Why? Let r be the smallest distance between the finitely many points. If you are a traveler that can only travel a distance $< r$, and you're at a point x, then you can not travel to any other point, and so we say that x is isolated.
    Can you make this intuitive explanation more formal? (don't use words like "travelers", instead, make short formal statements that cite definitions and/or theorems).

    Example 2: Say $M = [0,1] \bigcup \{2,3,4,5\}$. Now suppose that in any given trip you can only travel a distance $< r$ where say $r = 0.1$. Then you can travel from 0.35 to 0.4 in one trip. You can also travel from 0.35 to 0.48 if you take two trips. But you can not travel from 2 to any other point. So 2 is isolated. The same goes for the points 3, 4, 5. If you're at any one of those points, and if $r > 0$ is small enough, and if you can only travel distances $< r$, then you're stuck.
    Can you make this more formal?
    Let $M = [0,1] \bigcup \{2,3,4,5\}$ with the usual distance function.
    Give formal proof for: The isolated points in M are $2, 3, 4, 5$.

    (d) A sequence $x_1, x_2, \ldots$ in M can only converge to x when there is some N such that all $x_i = x$ for all $i \geq N$. In other words, when there is some tail $x_N, x_{N+1}, \ldots$ of your sequence that equals $x, x, \ldots$.

13. x is not isolated when

    (a) $\{x\}$ is not open.

    (b) Every neighborhood of x will contain more elements than just x.

    (c) For every $r > 0$ the set $S_r(x)$ contains more than just x.

    (d) There exists a sequence $x_1, x_2, \ldots$ in M that converges to x but where $x_n \neq x$ for every n
    (To produce such a sequence, do the following: for every n, the set $S_{\frac{1}{n}}(x) - \{x\}$ is not empty by part (c), so we can choose some x_n in $S_{\frac{1}{n}}(x) - \{x\}$. Then $x_n \neq x$ but $D(x_n, x) < \frac{1}{n}$ and therefore $x_1, x_2, \ldots$ converges to x.)

14. Let $x_1, x_2, \ldots$ be a sequence. A **tail** is what you get when you throw away the first ... (finitely many) elements. So a tail is a subsequence of the form $x_N, x_{N+1}, \ldots$ for some N (here we threw away the first $N-1$ elements).

15. $x_1, x_2, \ldots$ **converges** to x when

    (a) For every $\epsilon > 0$ the sequence has a tail contained in $S_\epsilon(x)$.

    (b) $\forall_{\epsilon>0}\, \exists_N\, \forall_{i \geq N}\, D(x_i, x) < \epsilon$

    When these equivalent properties hold then we say that x is the limit of the sequence $x_1, x_2, \ldots$.
    The most boring convergent sequences are those that have a tail that is constant (meaning: sequences for which there exists some N such that all the x_i with $i \geq N$ are the same). Such a sequence obviously converges. If x is isolated, then item 12(d) says that only boring sequences can converge to x.
    However, if x is not isolated, then there are more interesting sequences that converge to x, see item 13(d).

16. M is a discrete set when

    (a) Every x in M is isolated.

    (b) $\{x\}$ is open for every $x \in M$.

    (c) Every set $U \subseteq M$ is open.
    To show how (b) implies (c), lets take some set U. Now for every $x \in U$ take the set $\{x\}$ (you might have infinitely many of these sets because U could perhaps be an infinite set). Now take the union of all those sets and you get

    $$\bigcup_{x \in U} \{x\}$$

    By item 10 this union is an open set. But clearly, this union is just U itself.

    If M is discrete then every $U \subseteq M$ is open, so if M is discrete then it makes little sense to talk about open sets. As an example, if M has finitely many elements, then M is discrete, see Example 1 in item 12.

17. A set $F \subseteq M$ is closed when
    (a) If there is a sequence $x_1, x_2, \ldots$ in F that converges to $x \in M$ then this x must be in F.
    (b) If $S_r(x) \bigcap F$ is not empty for every $r > 0$ then $x \in F$.
    (c) If $F \bigcap U \neq \emptyset$ for every neighborhood U of x then $x \in F$.
    (d) If every neighborhood of x intersects F (if every neighborhood of x has element(s) in common with F) then $x \in F$.
    (e) The complement of F is open, i.e. $F^c = M - F$ is open.
    (f) F contains all of its limit points (if x is a limit point of F then x is in F).
18. A point x is called a *limit point* of A when there is a sequence in $A - \{x\}$ that converges to x.
19. Notation: $\overline{A}$ is called the *closure* of the set A
    (a) $\overline{A}$ is the union of A and all of its limit points.
    (b) $\overline{A}$ is the smallest closed set that contains A.
    (c) $\overline{A}$ is the intersection of all closed sets that contain A.
    (d) $x \in \overline{A} \iff$ every neighborhood of x intersects A.
    (e) $x \in \overline{A} \iff \exists$ a sequence $x_1, x_2, \ldots \in A$ that converges to x.
    (f) $x \in \overline{A} \iff \forall_{\epsilon > 0}$ there is a point in A that is ϵ-close to x.
20. x is a *limit point* of A if x is in the closure of $A - \{x\}$.
21. If $x_1, x_2, \ldots$ converges to x and $y_1, y_2, \ldots$ converges to y, then $D(x_1, y_1), D(x_2, y_2), \ldots$ converges to $D(x, y)$.
    This is Theorem 37, the proof uses the triangle inequality.
    Use this fact to prove exercises 6(a), 6(b), and 6(c) on page 78.
22. The diameter of a set A is the supremum of $\{D(x, y) | x, y \in A\}$.
23. If A is a set, then the diameter of A equals the diameter of $\overline{A}$. To prove this, you need item 21.
24. The union of *finitely many* closed sets is again closed.
25. The intersection of closed sets (even if you take infinitely many closed sets!) is again closed.
26. Exercise: Prove that a set with one point is closed. Then, using item 24 it follows that every finite set is closed.
27. A subset $A \subseteq M$ is called *dense* in M if
    (a) $\overline{A} = M$
    (b) For every $x \in M$ there exists a sequence $x_1, x_2, \ldots \in A$ that converges to x (see 19e)
    (c) Every non-empty open set intersects A (see 19d)