List of facts on cardinal numbers. We will prove some of these in class.

1. $o(A) = o(B)$ means $\exists f : A \to B$ with f bijection.
2. $o(A) \leq o(B)$ means $\exists f : A \to B$ with f one-to-one.
3. $\aleph_0$ is short notation for $o(\mathbb{N}^*)$.
4. c is short notation for $o(\mathbb{R})$.
5. The set A is *countably infinite* when: $o(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with f bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of A fit into one sequence $f(1), f(2), \ldots$.
6. Notation: $x < y$ is short for: $x \leq y \wedge x \neq y$.
7. $o(A) < o(P(A))$.
   By item 6 we should prove: $o(A) \leq o(P(A))$, and $o(A) \neq o(P(A))$.
   $o(A) \leq o(P(A))$ because there is a 1-1 function $f : A \to P(A)$
   (if $a \in A$ then $\{a\} \in P(A)$, so you can define the function $f(a) = \{a\}$, and this function is 1-1).
   We will prove $o(A) \neq o(P(A))$ by showing that there is no onto function from A to $P(A)$. Take an arbitrary function $f : A \to P(A)$. To prove: f is not onto. To prove: $\exists b \in P(A)$ for which $\forall_{a \in A}\ b \neq f(a)$.
   Proof: Take $b = \{a \in A | a \notin f(a)\}$. Then $\forall_{a \in A}\ b \neq f(a)$ because......
8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $o(\mathbb{N}^*) = \aleph_0$ is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!
9. If $f : A \to B$ is onto then $o(B) \leq o(A)$.
   To prove this, we need to accept the axiom of choice, i.e., we'd have to accept arguments that involve infinitely many choices. Then the proof goes like this: when f is onto, then for every $b \in B$ there exists at least one $a \in A$ with $f(a) = b$. For each b, choose one of the corresponding a's and then let $g(b)$ be that a. Then $g : B \to A$ is 1-1 and we get $o(B) \leq o(A)$.
10. A is *countable* when either: A is countably infinite (defined in item 5) or A is finite.
11. A is countable when $o(A) \leq \aleph_0$.
12. A subset of a countable set is again countable (page 22).

13. If $A \subseteq B$ then $o(A) \leq o(B)$. Proof: if $A \subseteq B$ then the function $x \mapsto x$ is a 1-1 function from A to B.
Note: items 11+13 imply item 12, because if A is a subset of some countable set B, then $o(A) \leq o(B)$ by item 13 and $o(B) \leq \aleph_0$ by item 11, hence $o(A) \leq \aleph_0$, and hence A is countable by item 11.

14. The ordering $\leq$ on cardinal numbers is a *partial ordering*.
In particular: whenever $d \leq e$ and $e \leq d$ we may conclude $d = e$.
The proof is not easy! (Theorem 9 in the book) (Schröder-Bernstein theorem).

15. The ordering $\leq$ on cardinal numbers is a *total ordering* (a chain), provided that one accepts the axiom of choice, but that is standard in math.
So given any two cardinals d, e we have $d \leq e$ or $d \geq e$. Another way to say this is that precisely one of these things must be true:
$d < e$, or $d = e$, or $d > e$ (of course, we already knew this for finite cardinals $0, 1, 2, 3, \ldots$ but it is not obvious (Theorem 11) that this should also be true for cardinal numbers).

16. Set A is uncountable when $o(A) \not\leq \aleph_0$. Using item 15 we can reformulate this by saying: A is uncountable when $o(A) > \aleph_0$.

17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).

18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

19. If you have countably many sets, and if each of these sets is countable, then their union is also countable (Theorem 1 in the book).

20. $\mathbb{R}$ is uncountable. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*))$.

21. If $d = o(D)$ and $e = o(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$. Likewise, $d \cdot e$ is the cardinality of $D \times E$.
d^e is the cardinality of D^E where $D^E = \{\text{all functions from } E \text{ to } D\}$.

22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same! (see Theorem 14 and 16).

23. There is a bijection between $P(A)$ and $\{0, 1\}^A$, and hence $o(P(A)) = o(\{0, 1\}^A) = o(\{0, 1\})^{o(A)} = 2^{o(A)}$.

To prove that such 1-1 correspondence exists I have to do the following: For any element of $P(A)$ I have to indicate what the corresponding element of $\{0,1\}^A$ is. So take an element $S \in P(A)$. To do: give the corresponding element of $\{0,1\}^A$. When $S \in P(A)$ then $S \subseteq A$, and we can now make the following function $f_S : A \to \{0,1\}$ as follows: $f_S(x) = 1$ when $x \in S$ and $f_S(x) = 0$ when $x \notin S$. This f_S is a function from A to $\{0,1\}$ and that means, by the notation introduced in item 21, that f_S is an element of $\{0,1\}^A$. This way I've shown a correspondence between elements of $P(A)$ and elements of $\{0,1\}^A$, and it is easy to see that this correspondence is 1-1 and onto.

24. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*)) = o(\{0,1\}^{\mathbb{N}^*}) = 2^{o(\mathbb{N}^*)} = 2^{\aleph_0}$.
(We will show $c = 2^{\aleph_0}$ in class by showing that $2^{\aleph_0} \leq c \leq 10^{\aleph_0} \leq 16^{\aleph_0} = (2^4)^{\aleph_0} = 2^{4\aleph_0} = 2^{\aleph_0}$. I'll show the first two $\leq$ in class by giving injective functions. The last two equalities come from using equation (12) on page 45 and Theorem 16 on page 43).

25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$
(equations (10),(11),(12) on page 45).

26. If you have d sets, and each of these sets has cardinality e, and if A is the union of all those sets, then $o(A) \leq de$ (if the d sets are disjoint, then you may replace the $\leq$ by $=$). Now if d or e is infinite, and both are non-zero, then we can also replace de by max(d,e), see item 22.
So for example, if you have d sets, and if each of those sets has e elements, and if both d and e are less than c, then the union of all those sets combined still has less than c elements. That means that the only way you could possibly write $\mathbb{R}$ as a union of smaller sets is when at least one of these two things is true: (i) you have very many (at least c) of these smaller sets, or (ii) some of these smaller sets must have just as many elements as $\mathbb{R}$ itself.
To illustrate (i), take for every real number x the set $\{x\}$. Now you have very many sets, namely c sets, each of which has small cardinality (each has cardinality 1). More generally, if I wanted to write $\mathbb{R}$ as a union of smaller sets, and if I want each of those sets to be countable, then I will be forced to take very many (at least c) of those sets.
To illustrate (ii) take three sets $(-\infty, 0)$, $\{0\}$, $(0, \infty)$. Their union is also $\mathbb{R}$ but this time the number of sets I've used is less than c (I used only $d = 3$ sets). The above reasoning shows that the only way $\mathbb{R}$ could be a union of d sets is if either d is at least c, or, at least one of those d sets has cardinality c (and indeed, in my example, two out of those $d = 3$ sets have cardinality c).

27. So far we have encountered these increasing cardinals:

$$0,\ 1,\ 2,\ 3,\ \ldots\ \aleph_0,\ \ c = 2^{\aleph_0},\ \ 2^c,\ \ 2^{2^c},\ \ldots$$

and we can wonder if there are any cardinals in between. Specifically, the *continuum hypothesis* asks if there is a cardinal d with $\aleph_0 < d < c$.

If we only assume the standard axioms of set theory (which we do in almost all of math!) then it is impossible to prove or disprove the continuum hypothesis! (how can mathematicians possibly know this?) Perhaps one day the continuum hypothesis becomes decidable if mathematicians decide to accept more axiom(s). However, this is unlikely to happen because the current list of axioms is good enough for nearly all of math outside of set theory.

28. Sample exercises using the above material:

(a) Prove that there exists a bijection between the line $\mathbb{R}$ and the plane $\mathbb{R}^2$.
Proof: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ so this has cardinality $c \cdot c$, which equals c using item 22. Thus, $\mathbb{R}^2$ has the same cardinality as $\mathbb{R}$, and this means (according to item 1) that there exists a bijection between $\mathbb{R}$ and $\mathbb{R}^2$.
(alternative way to prove this: First prove 2.6 Ex 6. Then take $d = c$ and $e = 2$ and use 2.6 Ex 6.)
Note: this proves, using the material from the previous pages, that such a bijection exists. But that does not mean that it is easy to *find* such a bijection!
We can indeed (the proof of Theorem 7+8 helps a lot here) find a bijection between the line $\mathbb{R}$ and the plane $\mathbb{R}^2$ but you won't need to do that for the test.

(b) Prove that there exists a bijection between $P(\mathbb{R})$ and $\mathbb{R}^{\mathbb{R}}$. In other words, prove that there are just as many subsets of $\mathbb{R}$, as there are functions from $\mathbb{R}$ to $\mathbb{R}$.
Proof: $\mathbb{R}^{\mathbb{R}}$ has cardinality c^c. Using item 24 we can write $(c)^c = (2^{\aleph_0})^c$. Using item 25 we can rewrite that as $2^{\aleph_0 c}$. By item 22 we know that $\aleph_0 c = c$ and so we can simplify $2^{\aleph_0 c}$ to 2^c which equals $2^{o(\mathbb{R})}$. But that's the number of subsets of $\mathbb{R}$, see item 23.

(c) (2.6 Ex 4). If $e \geq \aleph_0$ and $2 \leq d \leq 2^e$, prove that $d^e = 2^e$.
Proof: $2^e \leq d^e$ because $2 \leq d$. If we can also prove that $d^e \leq 2^e$ then we can use item 14 to conclude that $d^e = 2^e$. Remains to prove: $d^e \leq 2^e$.
$d^e \leq (2^e)^e$ because $d \leq 2^e$. Now $(2^e)^e = 2^{e \cdot e} = 2^e$ using item 25,

and item 22 (and the fact that e is infinite). So we see $d^e \leq (2^e)^e = 2^e$ and we are done.

(d) (2.6 Ex 6). If d_1 and e are infinite cardinals, and if $d = 2^{d_1}$ and $d \geq 2^e$ then show that $d^e = d$.
Proof: e is infinite means that $e \geq \aleph_0$, so clearly $e \geq 1$ and hence $d^e \geq d^1 = d$. Remains to show that $d^e \leq d$ (because we may then conclude $d^e = d$ using item 14).
Remains to prove $d^e \leq d$. Now $d = 2^{d_1}$ so $d^e = (2^{d_1})^e = 2^{d_1 e} = 2^{ed_1} = (2^e)^{d_1} \leq d^{d_1} = (2^{d_1})^{d_1} = 2^{d_1 d_1} = 2^{d_1} = d$ (using item 25 and item 22, and using the assumption $d \geq 2^e$ to see that $(2^e)^{d_1} \leq d^{d_1}$).