Sample Final for Thursday's class

1. Give the definition of:
   (for (a)–(d) give the definition, not a statement about cardinalities)

   (a) A function $f : A \to B$ is called *injective* when:
   (b) A function $f : A \to B$ is called *surjective* when:
   (c) The *power set* $P(A)$ of a set A is:
   (d) The product $A \times B$ of two sets is the set of all:
   (e) The *contrapositive* of a statement $p \implies q$ is:

   Answer:

2. Let $f : A \to B$ and consider the following statement:

   $$S : \quad \exists_{b \in B} \, \forall_{a \in A} \; f(a) \neq b$$

   Compute $\neg S$ (the negation of S). What does $\neg S$ say about f?

3. Let $x \in \mathbb{R}$. Write down the *contrapositive* of the following statement:

   $$S : \quad (\forall_{\epsilon > 0} \; |x| < \epsilon) \implies x = 0.$$

   Is S true? (Prove or disprove).

4. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$

   For the last two, justify your answer by showing your steps.

   (a) $\mathbb{Q} - \mathbb{Z}$
   (b) $P(\mathbb{N})$
   (c) $\{2, 2\}$
   (d) $\mathbb{R}^{\mathbb{N}}$
   (e) $\mathbb{R}^{\mathbb{R}}$

5. For each of the following subsets of $\mathbb{R}$, mention if it is open, closed, both, or neither. For each set A that is not closed, write down its closure $\overline{A}$:

   $\emptyset$

   $[0, \infty)$

   $\mathbb{R} - \{0\}$

   $(0, 1) \bigcap \mathbb{Q}$

   $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \{1/n \mid n \in \mathbb{N}^*\}$

6. Suppose that A, B are infinite sets and that $f : P(A) \to A \times B$ is injective. Must there exist an injective function from $P(A)$ to B? (Prove or disprove).

7. Let M be a metric space, and let A and B be subsets of M.

   (a) Suppose that for every a in A and every $\epsilon > 0$ there exists b in B with $D(a, b) < \epsilon$. Then show that $A \subseteq \overline{B}$.

   (b) Suppose for every $a \in A$ there is a sequence in B that converges to a. Show that $\overline{A} \subseteq \overline{B}$. (hint: first show $A \subseteq \overline{B}$).

   (c) Suppose $p \notin \overline{A}$. Show that there exists $\epsilon > 0$ with $S_\epsilon(p) \bigcap A = \emptyset$.

8. Let S be the set $\mathbb{N} \times \mathbb{N}$.

   (a) We will use the following ordering on S:
   $(x_1, y_1) \leq (x_2, y_2)$ is true when: $y_1 < y_2$ or ($y_1 = y_2$ and $x_1 \leq x_2$).
   Is this is a well-ordering on S? If so, what is its order type (write it as an ordinal number).

   (b) This time we will use another ordering, now $(x_1, y_1) \leq (x_2, y_2)$ when:
   $x_1 + y_1 < x_2 + y_2$ or ($x_1 + y_1 = x_2 + y_2$ and $x_1 \leq x_2$).
   Is a well-ordering on S? If so, what is its order type (write it as an ordinal number).

   Hint for both: Draw the set S on scratch paper, one for part (a), and another for part (b), then pick some points and indicate what the next-smallest point is. This way you can visualize the ordering.