Sample questions, Intro Advanced Math

1. Simplify the following cardinal numbers to the point where they look like $0, 1, 2, \ldots$, or $\aleph_0$, or c, or 2^c, or $2^{(2^c)}$, etc.

   $o(\{3, 3, 3\})$

   $o(P(\emptyset))$

   $o(\mathbb{Q})$

   $o(\mathbb{R} \times \mathbb{N})$

   $\aleph_0^{\aleph_0}$

   $o(P(\mathbb{R}))$

   $o(\mathbb{R}^{\mathbb{N}})$

   Does there exist an injective function from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$?

2. For each of the following sets (in the metric space $\mathbb{R}$), mention if it is open, closed, both, or neither. Also, for each set A that is not closed, write down what its closure $\overline{A}$ is:

   $(0, 1) \bigcup \{2\}$

   $\mathbb{Q}$

   $\emptyset$

   $\mathbb{R} - \mathbb{Z}$.

3. Give the definitions:

   (a) A function $f : A \to B$ is **injective** (same as "one-to-one") when:

   (b) Sets A and B have the **same cardinality** when:

4. If $f : A \to B$ and $g : B \to C$ are both **injective** (exercise 3a) then prove that the composition $g \circ f : A \to C$ is also **injective**.

5. Using **only** the definition from exercise 3b, show that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ have the **same cardinality**.

6. Let A, B be sets and let $C = A \times B$. Show that **at least one** of these must be true:

   (1) C is a **finite** set.

   or

   (2) There exists a **bijection** from A to C.

   or

   (3) There exists a **bijection** from B to C.

7. Let M be a metric space and A be a subset of M. We say that p is an **interior point of** A if there exists an open set U such that $p \in U$ and $U \subseteq A$. Let A_{int} be the set of all interior points of A. Show that A_{int} is an open set (partial credit if you prove: A open $\implies A_{\text{int}} = A$).

8. Proving a calculus formula by induction. I'll write f and g instead of $f(x)$ and $g(x)$ to keep the notation short. Lets say the calculus book proved

   (a) $1' = 0$
   (b) $x' = 1$
   (c) and the product rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

   But lets say the book did not yet prove that $(x^n)' = nx^{n-1}$, it will be our task to prove that. As an example, lets try $n = 2$:
   We can compute $(xx)'$ with the product rule $x' \cdot x + x \cdot x' = 1x + x1 = 2x$.
   Now lets try $n = 3$:
   Compute $(x^3)' = (x^2 x)'$ with the product rule as $(x^2)' \cdot x + x^2 \cdot x' = (2x) \cdot x + x^2 \cdot 1 = 3x^2$.
   Do the same for $(x^4)' = (x^3 x)' = \ldots$ and we start to see a pattern.
   Now lets really prove that $(x^n)' = nx^{n-1}$ is true.
   Denote $L(n) = (x^n)'$ and $R(n) = nx^{n-1}$ and let $P(n)$ be the statement $L(n) = R(n)$ that we want to prove. Now use induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$.