Final, Intro Advanced Math, ANSWERS

1. Simplify the following cardinal numbers to the point where they look like $0, 1, 2, \ldots$, or $\aleph_0$, or c, or 2^c, or $2^{(2^c)}$, etc.

$o(\{3,3,3\})$	$= 1$
$o(P(\emptyset))$	$= 2^0 = 1$
$o(\mathbb{Q})$	$= \aleph_0$
$o(\mathbb{R} \times \mathbb{N})$	$= c \cdot \aleph_0 = c$
$\aleph_0^{\aleph_0}$	$= c$ (explanation: $2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0}$)
$o(P(\mathbb{R}))$	$= 2^c$
$o(\mathbb{R}^{\mathbb{N}})$	$= c$ (explanation: $c^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0}$)

   Does there exist an injective function from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$?
   Yes (both have cardinality c so there is even a bijection).

2. For each of the following sets (in the metric space $\mathbb{R}$), mention if it is open, closed, both, or neither. Also, for each set A that is not closed, write down what its closure $\overline{A}$ is:

$(0,1) \bigcup \{2\}$	Neither.	$[0,1] \bigcup \{2\}$.
$\mathbb{Q}$	Neither.	$\mathbb{R}$.
$\emptyset$	Both.	$\emptyset$
$\mathbb{R} - \mathbb{Z}$	Open.	$\mathbb{R}$.

3. Give the definitions:

   (a) A function $f : A \to B$ is **injective** (same as "one-to-one") when:

   $f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$ (for all $a_1, a_2 \in A$)

   (b) Sets A and B have the **same cardinality** when:

   When there exists a bijection from A to B.
   (Note: this is item (1) in List of facts for Chapter 2).

4. If $f : A \to B$ and $g : B \to C$ are both **injective** (exercise 3a) then prove that the composition $g \circ f : A \to C$ is also **injective**.

   Let $a_1, a_2 \in A$ and assume $g(f(a_1)) = g(f(a_2))$. To prove: $a_1 = a_2$.
   Given: g is injective so $g(b_1) = g(b_2) \Longrightarrow b_1 = b_2$ for any $b_1, b_2 \in B$.
   Applying that to $b_1 = f(a_1),\ b_2 = f(a_2)$ gives $f(a_1) = f(a_2)$.
   Then the statement in Exercise 3(a) gives $a_1 = a_2$.

5. Using **only** the definition from exercise 3b, show that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ have the **same cardinality**.

   To prove: there exists a bijection from $\mathbb{N}$ to $\mathbb{N}^*$.
   Proof: take the function $f(n) := n + 1$.

6. Let A, B be sets and let $C = A \times B$. Show that **at least one** of these must be true:
   (1) C is a **finite** set.
   or (2) There exists a **bijection** from A to C.
   or (3) There exists a **bijection** from B to C.

   If not (1), then C is infinite and $o(C) = o(A)o(B) = \max(o(A), o(B))$ by item 22. Then $o(C)$ equals $o(A)$ or $o(B)$, hence (2) or (3).

7. Let M be a metric space and A be a subset of M. We say that p is an **interior point of** A if there exists an open set U such that $p \in U$ and $U \subseteq A$. Let A_{int} be the set of all interior points of A. Show that A_{int} is an open set (partial credit if you prove: A open $\Longrightarrow A_{\text{int}} = A$).

   Let $p \in A_{\text{int}}$.
   To prove (see items 6 and 5(c)): A_{int} contains a neighborhood of p.
   The exercise tells us that A contains an open subset U with $p \in U$. This U is a neighborhood of p (see item 9(b)). Remains to prove: $U \subseteq A_{\text{int}}$. Let $x \in U \subseteq A$. To prove: $x \in A_{\text{int}}$. Since U is open, x meets the definition of "interior point" and so $x \in A_{\text{int}}$.

8. Given:

   (a) $1' = 0$
   (b) $x' = 1$
   (c) and the product rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

   Statement $P(n)$ says $(x^n)' = nx^{n-1}$. Prove $P(n)$ for all $n \in \mathbb{N}$.
   Watch video "Proofs by Induction" at time 34:00, it says we have to do two things

   - Step 1: Verify $P(0)$: $(x^0)' = 0x^{0-1}$.
     True, the left-hand-side $(x^0)' = 1'$ is 0 by (a).
   - Step 2: Prove $P(n) \Longrightarrow P(n+1)$ for all n.
     Assume $P(n)$. To prove: $P(n+1)$, which says $(x^{n+1})' = (n+1)x^n$.
     Proof: $(x^{n+1})' = (x^n x)' =$ [use (c)]
     $= (x^n)'x + x^n x' =$ [use $P(n)$] $= nx^{n-1}x + x^n x' =$ [use (b)]
     $= nx^{n-1}x + x^n = nx^n + x^n = (n+1)x^n$.