Answers to Test 3 Ex 1. April 16, 2020.

1. In Ex 1 you do not need to give proofs/examples/explanations:

   (a) In the metric space $M = \mathbb{Z}$ take the set $A = \{0\}$.

   i. A is open in M: True/False

   Answer in 4.2 video #1 at time 21:00.
   Also in 4.2 video #3 at time 5:00.

   ii. A is closed in M: True/False

   Answer in video 4.3 #1 time 15:40.

   (b) In the metric space $M = \mathbb{R}$ take the set $A = \{0\}$.

   i. A is open in M: True/False

   Answer in video 4.2 #3 at time 9:30.
   Also in video "Help with HW from 4.1 and 4.2" at time 25:00.
   Also in video "Help for second set of HW for Section 4.2" at time 21:30.

   ii. A is closed in M: True/False

   Answer in video 4.3 #1 time 15:40

   (c) Let $M = \mathbb{R}$ and $A \subseteq M$ and suppose that

   $$\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\} \subseteq A$$

   and $0 \notin A$. Then

   i. A is closed in M: True/False/NI

   Answer in video 4.3 #3 time 14:05 (different example but same proof)
   Also in "video to answer to the sample test questions related to section 4.3" at time 6:50.

   ii. A is open in M: True/False/NI

   NI: Such A could be open (e.g. $\mathbb{R}$), or not open (e.g. $\mathbb{Q}$).
   For examples see:
   Video 4.2 #1 at time 39:00.
   Video 4.2 #3 at time 18:30.

   Note: NI means "Not enough Information was given to decide for certain" (in other words: there are examples where it's True, and there are examples where it's False).

   (d) Let M be any metric space, and let $A = M$.
   Then A is closed in M: True/False/NI.

   Answer in video 4.3 #1 at time 32:20.

(e) Let M be a metric space, A a subset of M. Suppose that A is not open. Then A is closed: True/False/NI.

Answer in many videos:
4.3 #1 time 15:40
"... sample test questions related to section 4.2" at time 8:00
"... sample test questions related to section 4.3" right at the beginning. Same video at time 20:40.

2. Let M be a metric space, and suppose that the sequence $x_1, x_2, x_3, \ldots$ converges to x. Suppose that U is open and that $x \in U$.
Prove that $U \bigcap \{x_1, x_2, x_3, \ldots\} \neq \emptyset$.

Proof 1: $x \in U$ and U open, so from the definition of open (in item 6 and 5(a)) there is some $r > 0$ with $S_r(x) \subseteq U$. Then from the definition of "converges" (in item 15(a)) the sequence has a tail (i.e. something that looks like $x_N, x_{N+1}, x_{N+2}, \ldots$ for some N) inside $S_r(x) \subseteq U$. But that tail is also inside $\{x_1, x_2, x_3, \ldots\}$, so both U and $\{x_1, x_2, x_3, \ldots\}$ contain that tail, and so the intersection is $\neq \emptyset$.

Proof 2: You can also prove this by contradiction: if you assume that $U \bigcap \{x_1, x_2, x_3, \ldots\} = \emptyset$ (then T.P. a contradiction) then $x_1, x_2, \ldots$ is in the complement of U. Now this complement $F := U^c$ is closed (see 17(e)) so then we can use the definition of closed (in item 17(a)) to conclude $x \in U^c$ which contradicts $x \in U$.

3. Let M be a metric space and let $a \in M$.
Let $U = \{x \in M \mid D(a, x) > 1\}$. Prove that U is open.

(Note: a is a fixed point, and U is the set of all points in M that have distance > 1 from that point a)

A similar question is found in video 4.2 #1 at time 51:00,
and in video 4.2 #2 at time 28:30, and in
"... sample test questions related to section 4.2" at time 22:10 and 25:00.

Proof: Following the definition of open, we have to do this:
Let $x \in U$, which means $D(a, x) > 1$. To prove: $\exists_{r>0}\ S_r(x) \subseteq U$.
Proof: Take $r := D(a, x) - 1$.

(Why does this work? Well, any point in $S_r(x)$ has distance $< r$ from x, so it has distance $> D(a, x) - r$ from a by the triangle inequality. So it has distance > 1 from a and is thus in U. So any point in $S_r(x)$ is in U).

List of facts for Chapter 4, shortened version for use with test 3.

1. A **metric space** M is set with a distance function with the following properties (for all $a, b, c \in M$): $D(a, a) = 0$, $D(a, b) > 0$ whenever $a \neq b$, $D(b, a) = D(a, b)$, and the triangle inequality: $D(a, c) \leq D(a, b) + D(b, c)$.
2. $S_r(x)$ is the **open ball** with radius r and center x.
$S_r(x) = \{p \in M | D(x, p) < r\}$. So this is the set of all points you can reach if you start from x and then travel a distance that is *less than* r.

3. We say that p and x are **r-close** when $D(p, x) < r$.
   So $S_r(x)$ is the set of all points that are r-close to x.
4. Any set that contains $S_r(x)$ for some $r > 0$ is called a **neighborhood** of x. So a set U is a neighborhood of x when there exists some positive r such that all points that are r-close to x are in the set U.
5. Let U be a subset of M. The following statements are **equivalent**:
   (a) $\exists_{r>0}\ S_r(x) \subseteq U$
   (b) U is a neighborhood of x
   (c) U contains a neighborhood of x.
6. A set $U \subseteq M$ is **open** when property 5(a)(b)(c) is true for every x in U.
7. Note: a neighborhood of x is **not the same** as an open set, because if we want to check that U is an open set then we need to check property 5(a) for *every* element of U. Whereas to check if U is a neighborhood of x, we only have to check property 5(a) for one element (namely x).
8. The sets $\emptyset$ and M are **always open** (even if M does not "look" open. To understand this, selecting M means selecting *all points* to be considered. Then all r-close points to any x in M are automatically in M).
9. An **open neighborhood** is (these conditions are equivalent):
   (a) A neighborhood of x that happens to be an open set.
   (b) An open set that happens to contain x.
10. **Any** union of open sets is always open (even infinitely many sets!).
11. The intersection of **finitely many** open sets is again open.
12. x is an **isolated point** when:
    (a) $\{x\}$ is open
    (b) There is a neighborhood of x that contains just x and no other elements.
    (c) $\exists_{r>0}\, S_r(x) = \{x\}$
    (d) A sequence $x_1, x_2, \ldots$ in M can only converge to x when there is some N such that all $x_i = x$ for all $i \geq N$. In other words, when there is some tail $x_N, x_{N+1}, \ldots$ of your sequence that equals $x, x, \ldots$.
13. x is **not isolated** when
    (a) $\{x\}$ is not open.
    (b) Every neighborhood of x will contain more elements than just x.
    (c) For every $r > 0$ the set $S_r(x)$ contains more than just x.
    (d) There exists a sequence $x_1, x_2, \ldots$ in M that converges to x but where $x_n \neq x$ for every n
    (To produce such a sequence, do the following: for every n, the set $S_{\frac{1}{n}}(x) - \{x\}$ is not empty by part (c), so we can choose some x_n in $S_{\frac{1}{n}}(x) - \{x\}$. Then $x_n \neq x$ but $D(x_n, x) < \frac{1}{n}$ and therefore $x_1, x_2, \ldots$ converges to x.)
14. Let $x_1, x_2, \ldots$ be a sequence. A **tail** is what you get when you throw away the first ... (finitely many) elements. So a tail is a subsequence of the form $x_N, x_{N+1}, \ldots$ for some N (here we threw away the first $N - 1$ elements).

15. $x_1, x_2, \ldots$ **converges** to x when

    (a) For every $\epsilon > 0$ the sequence has a tail contained in $S_\epsilon(x)$.

    (b) $\forall_{\epsilon>0}\, \exists_N\, \forall_{i \geq N}\, D(x_i, x) < \epsilon$

    When these equivalent properties hold then we say that x is the limit of the sequence $x_1, x_2, \ldots$.
    The most boring convergent sequences are those that have a tail that is constant. Such a sequence obviously converges. If x is isolated, then item 12(d) says that only boring sequences can converge to x.
    However, if x is not isolated, then there are more interesting sequences that converge to x, see item 13(d).

16. M is **discrete** when

    (a) Every x in M is isolated.

    (b) $\{x\}$ is open for every $x \in M$.

    (c) Every set $U \subseteq M$ is open.

17. A set $F \subseteq M$ is **closed** when

    (a) If a sequence $x_1, x_2, \ldots$ in F converges to x then x must be in F.

    (b) If $S_r(x) \bigcap F$ is not empty for every $r > 0$ then $x \in F$.

    (c) If $F \bigcap U \neq \emptyset$ for every neighborhood U of x then $x \in F$.

    (d) If every neighborhood of x intersects F (if every neighborhood of x has element(s) in common with F) then $x \in F$.

    (e) The complement of F is open, i.e. $F^c = M - F$ is open.

    (f) F contains all of its limit points (x is a limit point of $F \Longrightarrow x \in F$).

18. A point x is called a **limit point** of A if there is a sequence in $A - \{x\}$ that converges to x.

19. $\overline{A}$ is called the **closure** of the set A.

    (a) $\overline{A}$ is the union of A and all of its limit points.

    (b) $\overline{A}$ is the smallest closed set that contains A.

    (c) $\overline{A}$ is the intersection of all closed sets that contain A.

    (d) $x \in \overline{A} \Longleftrightarrow$ every neighborhood of x intersects A.

    (e) $x \in \overline{A} \Longleftrightarrow \exists$ a sequence $x_1, x_2, \ldots \in A$ that converges to x.

    (f) $x \in \overline{A} \Longleftrightarrow \forall_{\epsilon>0}$ there is a point in A that is ϵ-close to x.

20. x is a **limit point** of A if x is in the closure of $A - \{x\}$.

21. If $x_1, x_2, \ldots$ converges to x and $y_1, y_2, \ldots$ converges to y, then $D(x_1, y_1), D(x_2, y_2), \ldots$ converges to $D(x, y)$.

22. The diameter of a set A is the supremum of $\{D(x, y) | x, y \in A\}$.

23. If A is a set, then the diameter of A equals the diameter of $\overline{A}$. To prove this, you need item 21.

24. The union of *finitely many* closed sets is again closed.

25. The intersection of closed sets (even if you take infinitely many closed sets!) is again closed.