Intro Advanced Math., test 2, March 10, 2020

1. For each, simplify the cardinality to one of: $0, 1, 2, \ldots$, $\aleph_0$, c, 2^c, 2^{2^c}, ...
   For (a)–(g) you do not need to show your work, but for (h),(i),(j) you need to justify your answer by showing all steps.

   (a) $P(\emptyset)$
   (b) $\{2, 2, 2\} - \{2\}$
   (c) $\mathbb{N}$
   (d) $\emptyset \times \mathbb{R}$
   (e) $P(\mathbb{R} \times \mathbb{Q})$
   (f) $P(\mathbb{Q} - \mathbb{Z})$
   (g) $P(\mathbb{R})$
   (h) $\mathbb{R}^{\mathbb{R}}$
   (i) $\mathbb{R}^{\mathbb{N}}$
   (j) $\mathbb{N}^{\mathbb{R}}$

2. Does there exist: (for some of these you can use Ex 1) (it suffices to write yes/no):

   (a) an injective function from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{N}^{\mathbb{R}}$?
   (b) an injective function from $\mathbb{N}^{\mathbb{R}}$ to $\mathbb{R}^{\mathbb{N}}$?
   (c) an injective function from $\mathbb{R}^{\mathbb{N}}$ to $P(\mathbb{N})$?
   (d) an injective function from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{N}$?

3. Let A be an infinite set, let $B \subseteq A$, and suppose there is no bijection from A to B.
   Must there then be a bijection from A to $A - B$? Yes/No. Give a proof for your answer.
   Label each piece of given information as (1), (2), (3) and make sure your proof uses each.

4. Suppose that A is an infinite set. Must there exist an injective function from $A \times \mathbb{Q}$ to A? Yes/No. Give a proof for your answer.

5. Suppose that $A \subseteq \mathbb{R}$ is countable. Must there exist $r \in \mathbb{R}$ such that $r \neq a + b$ for all $a \in A$ and all $b \in \mathbb{Q}$? Explain. Hint: Ex 4.

6. Suppose that there exists an injective function from $\mathbb{N}$ to $P(A)$.
   Prove that there is an injective function from $\mathbb{R}$ to $P(A)$.

If you run out of time then mark one of Ex 3-6 as HW.

List of facts on cardinal numbers, shortened version.

1. $o(A) = o(B)$ means $\exists f : A \to B$ with f bijection.

2. $o(A) \leq o(B)$ means $\exists f : A \to B$ with f one-to-one.

3. $\aleph_0$ is short notation for $o(\mathbb{N}^*)$.

4. c is short notation for $o(\mathbb{R})$.

5. The set A is *countably infinite* when: $o(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with f bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of A fit into one sequence $f(1), f(2), \ldots$.

6. Notation: $x < y$ is short for: $x \leq y \wedge x \neq y$.
7. $o(A) < o(P(A))$.
8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $o(\mathbb{N}^*) = \aleph_0$, is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!
9. If $f : A \to B$ is onto then $o(B) \leq o(A)$.
10. A is *countable* when either: A is countably infinite (defined in item 5) or A is finite.
11. A is countable when $o(A) \leq \aleph_0$.
12. A subset of a countable set is again countable.
13. If $A \subseteq B$ then $o(A) \leq o(B)$.
14. The ordering $\leq$ on cardinal numbers is a *partial ordering*.
    In particular: whenever $d \leq e$ and $e \leq d$ we may conclude $d = e$.
    You might remember that the proof was not easy!
15. The ordering $\leq$ on cardinal numbers is a *total ordering*. So given any two cardinals d, e we have $d \leq e$ or $d \geq e$. This means that one of these things must be true: $d < e$ or $d = e$ or $d > e$.
16. Set A is uncountable when $o(A) \not\leq \aleph_0$. Using item 15 we can reformulate this by saying: A is uncountable when $o(A) > \aleph_0$.
17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).
18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.
19. If you have countably many sets, and if each of these sets is countable, then their union is also countable.
20. $\mathbb{R}$ is uncountable. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*))$.
21. If $d = o(D)$ and $e = o(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$.
    Likewise, $d \cdot e$ is the cardinality of $D \times E$.
    d^e is the cardinality of D^E where $D^E = \{\text{all functions from } E \text{ to } D\}$.
22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same!
23. There is a bijection between $P(A)$ and $\{0,1\}^A$, and hence $o(P(A)) = o(\{0,1\}^A) = o(\{0,1\})^{o(A)} = 2^{o(A)}$.
24. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*)) = o(\{0,1\}^{\mathbb{N}^*}) = 2^{o(\mathbb{N}^*)} = 2^{\aleph_0}$.
25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$
26. If you have d sets, and each of these sets has cardinality e, and if A is the union of all those sets, then $o(A) \leq de$ (if the d sets are disjoint, then you may replace the $\leq$ by $=$). Now if d or e is infinite, and both are non-zero, then we can also replace de by max(d,e), see item 22.