Intro Advanced Math., test 2 ANSWERS, March 10, 2020

1. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$
   For (a)–(g) you do not need to show your work, but for (h),(i),(j) you need to justify your answer by showing all steps.

   (a) $P(\emptyset)$: 1
   (b) $\{2,2,2\} - \{2\}$: 0
   (c) $\mathbb{N}$: $\aleph_0$
   (d) $\emptyset \times \mathbb{R}$: 0
   (e) $P(\mathbb{R} \times \mathbb{Q})$: 2^c
   (f) $P(\mathbb{Q} - \mathbb{Z})$: c
   (g) $P(\mathbb{R})$: 2^c
   (h) $\mathbb{R}^{\mathbb{R}}$: $c^c = (2^{\aleph_0})^c = 2^{\aleph_0 c} = 2^c$
   (i) $\mathbb{R}^{\mathbb{N}}$: $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = c$
   (j) $\mathbb{N}^{\mathbb{R}}$: $2^c \leq \aleph_0^c \leq c^c =$ [Ex 1(h)] $= 2^c$ so $\aleph_0^c = 2^c$ by item 15.

2. Does there exist: (for some of these you can use Ex 1) (it suffices to write yes/no):

   (a) an injective function from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{N}^{\mathbb{R}}$? Yes
   (b) an injective function from $\mathbb{N}^{\mathbb{R}}$ to $\mathbb{R}^{\mathbb{N}}$? No
   (c) an injective function from $\mathbb{R}^{\mathbb{N}}$ to $P(\mathbb{N})$? Yes
   (d) an injective function from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{N}$? Yes

3. Let A be an <u>infinite set (1)</u>, let <u>$B \subseteq A$ (2)</u>, and suppose there is <u>no bijection from A to B (3)</u>. Must there then be a bijection from A to $A - B$? Yes/No. Give a proof for your answer.

   From (2): $B \subseteq A$ so $A = B \bigcup (A - B)$ and hence $o(A) = o(B) + o(A - B)$.
   Then from (1) and item 22: $o(A) = \max(o(B), o(A - B))$.

   Now $o(A) = \max(o(B), o(A - B))$ but $o(A) \neq o(B)$ from (3) and so $o(A) = o(A - B)$ which means a bijection from A to $A - B$ exists.

4. Suppose that A is an infinite set. Must there exist an injective function from $A \times \mathbb{Q}$ to A?

   From item 22 and A infinite: $o(A \times \mathbb{Q}) = o(A)o(\mathbb{Q}) = o(A)\aleph_0 = \max(o(A), \aleph_0) = o(A)$.
   (the last = is because $o(A) \geq \aleph_0$ for any infinite set A). So yes there must be a bijection.

5. Suppose that $A \subseteq \mathbb{R}$ is countable. Must there exist $r \in \mathbb{R}$ such that $r \neq a + b$ for all $a \in A$ and all $b \in \mathbb{Q}$? Explain. Hint: Ex 4 or item 19.

   Proof 1: $S := \{a + b \mid a \in A, b \in \mathbb{Q}\} = \bigcup_{b \in \mathbb{Q}} A + b$ is a countable union of countable sets, and hence countable. But $\mathbb{R}$ is not countable so it must have an element r that is not in S.
   Proof 2: $A \times \mathbb{Q}$ is countable so no function $A \times \mathbb{Q} \to \mathbb{R}$ can be onto. So the function that sends $(a, b) \in A \times \mathbb{Q}$ to $a + b$ can not be onto. Therefore there exists $r \in R$ that is not of the form $a + b$ with $a \in A$ and $b \in \mathbb{Q}$.

6. Suppose that there exists an injective function from $\mathbb{N}$ to $P(A)$.
   Prove that there is an injective function from $\mathbb{R}$ to $P(A)$.

   The injective function shows that $P(A)$ is infinite. Then A is infinite, so $o(A) \geq \aleph_0$ and $o(P(A)) = 2^{o(A)} \geq 2^{\aleph_0} = c$. This $o(P(A)) \geq c$ means there is an injective function $\mathbb{R} \to P(A)$.