Intro Advanced Math., test 2 answers, October 21, 2015

1. $\aleph_0 + \aleph_0 = \aleph_0$

   $o(\{3,3,3\}) = 1$

   $o(P(\emptyset)) = 2^0 = 1$

   $o(P(\mathbb{R})^{\mathbb{R}}) = (2^c)^c = 2^{cc} = 2^c$

   $o(\mathbb{R} \times \mathbb{N}) = c\aleph_0 = c$

   $7^{\aleph_0} = 2^{\aleph_0} = c$

   $\aleph_0 + c + 2^{\aleph_0} = c$

   $(2^c)^{c^2} = (2^c)^c = 2^{cc} = 2^c$

   $o(P(\mathbb{Q}) - P(\mathbb{N})) = c$

   $o(\emptyset^{\mathbb{R}} \times \mathbb{R}) = 0c = 0$

   $o(\mathbb{R}^{\mathbb{N}}) = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = c$

   $o(P(\mathbb{N})) = 2^{\aleph_0} = c$

   Note: $P(\mathbb{Q}) - P(\mathbb{N})$ contains $P(\mathbb{Q} - \mathbb{N}) - \{\emptyset\}$ which has cardinality $2^{o(\mathbb{Q}-\mathbb{N})} = 2^{\aleph_0} = c$.

2. Let A, B, C be sets.

   (a) Write down the definition of $o(A) \leq o(B)$.

   This means there exists an injective function $f : A \to B$.

   (b) We know that

   $$\text{if} \quad o(A) \leq o(B) \quad \text{and} \quad o(B) \leq o(C) \quad \text{then} \quad o(A) \leq o(C).$$

   Write down a proof for this rule using only the definition of $\leq$.

   If there is an injective function $f : A \to B$ and an injective function $g : B \to C$ then there is an injective function $A \to C$, namely, just take the composition $g \circ f$.

3. (a) Give the definition of cardinal addition: if $D \bigcap E = \emptyset$ and $d = o(D)$ and $e = o(E)$ then $d + e$ is defined as: $o(D \bigcup E)$.

   (b) Let A be an infinite set, let B be a subset and let $C = A - B$.
   Suppose that there is a bijection from B to C.
   Prove that there is a bijection from A to C.

   $o(A) = o(B) + o(C) = o(C) + o(C) = \max(o(C), o(C)) = o(C)$, hence there is a bijection from A to C.
   The first equality holds because $A = B \bigcup C$ and B, C are disjoint.
   The second equality holds because there is a bijection from B to C.
   The third equality holds because A (and hence C) is an infinite set.

   (c) Give the definition of cardinal multiplication: $de = o(D \times E)$.

   (d) Let A, B, C as in part (b), and assume that B and C are not empty.
   Prove that there is a bijection from A to $B \times C$.

   $o(B \times C) = o(B)o(C) = \max(o(B), o(C)) = o(C) = o(A)$. Hence there is a bijection from A to $B \times C$.

4. Let $\mathbb{N}^* = \{1, 2, 3, 4, \ldots\}$, $E = \{2, 4, 6, 8, \ldots\}$, $D = \{1, 3, 5, 7, \ldots\}$.
So $E = \{\text{all even positive integers}\}$, and $D = \{\text{all odd positive integers}\}$.

   (a) Give a bijection $f : \mathbb{N}^* \to E$
   $f(n) = 2n$ (note: this proves $o(\mathbb{N}^*) = o(E)$)

   (b) Give a bijection $g : \mathbb{N}^* \to D$.
   $g(n) = 2n - 1$ (note: this proves $o(\mathbb{N}^*) = o(D)$)

   (c) Explain why parts (a),(b), and Ex.3(a), show: $\aleph_0 + \aleph_0 = \aleph_0$.
   $\mathbb{N}^*$ is the disjoint union of D, E and hence
   $\aleph_0 = o(\mathbb{N}^*) = o(D \bigcup E) = o(D) + o(E) = \aleph_0 + \aleph_0$.

5. Take home: Suppose that for each $n \in \mathbb{N}^*$ you are given a subset $A_n \subseteq \mathbb{R}$. Suppose that $\mathbb{R} = A_1 \bigcup A_2 \bigcup A_3 \bigcup \cdots$. Show that at least one of those sets A_n must be uncountable.

   If all of the A_n are countable, then $A_1 \bigcup A_2 \bigcup A_3 \bigcup \cdots$ is a countable union of countable sets, hence countable. But $\mathbb{R}$ is not countable, so not all A_n can be countable.

6. Take home: Let $A = \{S \subset \mathbb{R} \mid S \text{ countable}\}$. So A is the set of all *countable* subsets of $\mathbb{R}$, in other words:
   $S \in A$ if and only if S is countable and $S \subset \mathbb{R}$.

   Prove that $o(A) = c$.
   Hint: Find an onto function $F : \mathbb{R}^{\mathbb{N}^*} \to B$ where $B = A - \{\emptyset\}$.

   An element $g \in \mathbb{R}^{\mathbb{N}^*}$ is a function $g : \mathbb{N}^* \to \mathbb{R}$. Now define:
   $F(g) := g(\mathbb{N}^*) = \{g(1), g(2), g(3), \ldots\} \in B$. This function F is onto, hence $o(B) \leq o(\mathbb{R}^{\mathbb{N}^*}) = c^{\aleph_0} = c$.

   The function $h : \mathbb{R} \to B$ with $h(x) = \{x\}$ is injective, hence
   $c = o(\mathbb{R}) \leq o(B)$. Combined, we find $o(B) = c$.
   Then $o(A) = o(B) + 1 = c + 1 = c$.