1. $\aleph_0 + \aleph_0 = \aleph_0$

   $\aleph_0 + c + 2^\aleph_0 = c$

   $\aleph_0 + 3, 3, 3\} = 1$

   $(2^\aleph_0)^2 = (2^\aleph_0)^2 = 2^{\aleph_0} = 2^c$

   $o(\emptyset) = 2^0 = 1$

   $o(\mathcal{P}(\emptyset)) = c$

   $o(\mathcal{P}(\mathbb{R})) = (2^\aleph_0)^c = 2^{\aleph_0} = 2^c$

   $o(\mathbb{R} \times \mathbb{N}) = c\aleph_0 = c$

   $7^{\aleph_0} = 2^{\aleph_0} = c$

   $2^\aleph_0 = (2^\aleph_0)^c = 2^{\aleph_0} = 2^c$

   $o(\mathcal{P}(\mathbb{R} \cup \emptyset)) = 0$,$^c$

   $o(\mathcal{P}(\mathbb{R} \times \mathbb{N})) = (2^\aleph_0)^{\aleph_0} = 2^{\aleph_0} \aleph_0 = \aleph_0 = c$

   $o(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0} = c$

   Note: $\mathcal{P}(\mathbb{Q}) - \mathcal{P}(\mathbb{N})$ contains $\mathcal{P}(\mathbb{Q} - \mathbb{N}) - \{\emptyset\}$ which has cardinality $2^\aleph_0 = \aleph_0 = c$.

2. Let $A, B, C$ be sets.

   (a) Write down the definition of $o(A) \leq o(B)$.

   This means there exists an injective function $f : A \rightarrow B$.

   (b) We know that

   \[\text{if } o(A) \leq o(B) \text{ and } o(B) \leq o(C) \text{ then } o(A) \leq o(C).\]

   Write down a proof for this rule using only the definition of $\leq$.

   If there is an injective function $f : A \rightarrow B$ and an injective function $g : B \rightarrow C$ then there is an injective function $A \rightarrow C$, namely, just take the composition $g \circ f$.

3. (a) Give the definition of cardinal addition: if $D \cap E = \emptyset$ and $d = o(D)$ and $e = o(E)$ then $d + e$ is defined as: $o(D \cup E)$.

   (b) Let $A$ be an infinite set, let $B$ be a subset and let $C = A - B$.

   Suppose that there is a bijection from $B$ to $C$.

   Prove that there is a bijection from $A$ to $C$.

   \[o(A) = o(B) + o(C) = o(C) + o(C) = \max(o(C), o(C)) = o(C), \text{ hence there is a bijection from } A \text{ to } C.\]

   The first equality holds because $A = B \cup C$ and $B, C$ are disjoint. The second equality holds because there is a bijection from $B$ to $C$. The third equality holds because $A$ (and hence $C$) is an infinite set.

   (c) Give the definition of cardinal multiplication: $de = o(D \times E)$.

   (d) Let $A, B, C$ as in part (b), and assume that $B$ and $C$ are not empty.

   Prove that there is a bijection from $A$ to $B \times C$.

   \[o(B \times C) = o(B)o(C) = \max(o(B), o(C)) = o(C) = o(A). \text{ Hence there is a bijection from } A \text{ to } B \times C.\]
4. Let \( \mathbb{N}^* = \{1, 2, 3, 4, \ldots\} \), \( E = \{2, 4, 6, 8, \ldots\} \), \( D = \{1, 3, 5, 7, \ldots\} \).

So \( E = \{\text{all even positive integers}\} \), and \( D = \{\text{all odd positive integers}\} \).

(a) Give a bijection \( f : \mathbb{N}^* \to E \)

\[ f(n) = 2n \]  
(note: this proves \( o(\mathbb{N}^*) = o(E) \))

(b) Give a bijection \( g : \mathbb{N}^* \to D \)

\[ g(n) = 2n - 1 \]  
(note: this proves \( o(\mathbb{N}^*) = o(D) \))

(c) Explain why parts (a),(b), and Ex.3(a), show: \( \aleph_0 + \aleph_0 = \aleph_0 \).

\( \mathbb{N}^* \) is the disjoint union of \( D, E \) and hence

\[ \aleph_0 = o(\mathbb{N}^*) = o(D \cup E) = o(D) + o(E) = \aleph_0 + \aleph_0. \]

5. Take home: Suppose that for each \( n \in \mathbb{N}^* \) you are given a subset \( A_n \subseteq \mathbb{R} \).

Suppose that \( \mathbb{R} = A_1 \cup A_2 \cup A_3 \cup \ldots \). Show that at least one of those sets \( A_n \) must be uncountable.

If all of the \( A_n \) are countable, then \( A_1 \cup A_2 \cup A_3 \cup \ldots \) is a countable union of countable sets, hence countable. But \( \mathbb{R} \) is not countable, so not all \( A_n \) can be countable.

6. Take home: Let \( A = \{S \subset \mathbb{R} \mid S \text{ countable}\} \). So \( A \) is the set of all countable subsets of \( \mathbb{R} \), in other words:

\( S \in A \) if and only if \( S \) is countable and \( S \subset \mathbb{R} \).

Prove that \( o(A) = c \).

Hint: Find an onto function \( F : \mathbb{R}^{\mathbb{N}^*} \to B \) where \( B = A - \{\emptyset\} \).

An element \( g \in \mathbb{R}^{\mathbb{N}^*} \) is a function \( g : \mathbb{N}^* \to \mathbb{R} \). Now define:

\( F(g) := \{ g(1), g(2), g(3), \ldots \} \in B \). This function \( F \) is onto, hence \( o(B) \leq o(\mathbb{R}^{\mathbb{N}^*}) = c^{\aleph_0} = c \).

The function \( h : \mathbb{R} \to B \) with \( h(x) = \{x\} \) is injective, hence \( c = o(\mathbb{R}) \leq o(B) \). Combined, we find \( o(B) = c \).

Then \( o(A) = o(B) + 1 = c + 1 = c \).