Test 3, Intro Advanced Math. April 15, 2019.

Four fully correct questions = 100 points.

1. Let M be a metric space and $x \in M$. Write down the negation of the following statement:

   $$\forall_{r>0} \exists_{p \in M} \quad p \neq x \wedge D(p,x) < r$$

   In words, what does your answer say? (look at List of Facts to get a short description).

2. For each of the following sets in the metric space $M = \mathbb{R}$, mention if it is open, closed, both, or neither. For each set A that is not closed, write down its closure $\overline{A}$:

   $\emptyset$

   $[0, \infty)$

   $\mathbb{Q} - \{0\}$

   $\mathbb{R} - \{0\}$

   $(0, 1]$

   $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \{1/n \mid n \in \mathbb{N}^*\}$

3. Let M be a metric space, A a closed subset, and p an isolated point. Prove that $A - \{p\}$ is closed.

4. Let M be a metric space, A is a subset of M, and $x \in M$. Suppose that $x \in \overline{A}$ and $x \notin A$. Show that x is not isolated.

5. Let M be a metric space, and let A be a subset of M. For $x \in M$ we say that the distance from x to A is less than 1 if there exists some $a \in A$ with $D(a, x) < 1$. Let U be the set of points in M that have distance less than 1 to A. In other words

   $$U = \{x \in M \mid \exists_{a \in A} \quad D(a,x) < 1\}$$

   Show that U is an open subset of M.

List of facts for Chapter 4, shortened version for use with test 3.

1. A **metric space** M is set with a distance function with the following properties (for all $a, b, c \in M$): $D(a,a) = 0$, $D(a,b) > 0$ whenever $a \neq b$, $D(b,a) = D(a,b)$, and the triangle inequality: $D(a,c) \leq D(a,b) + D(b,c)$.
2. $S_r(x)$ is the **open ball** with radius r and center x.
   $S_r(x) = \{p \in M | D(x,p) < r\}$. So this is the set of all points you can reach if you start from x and then travel a distance that is *less than* r.
3. We say that p and x are **r-close** when $D(p,x) < r$.
   So $S_r(x)$ is the set of all points that are r-close to x.
4. Any set that contains $S_r(x)$ for some $r > 0$ is called a **neighborhood** of x. So a set U is a neighborhood of x when there exists some positive r such that all points that are r-close to x are in the set U.
5. Let U be a subset of M. The following statements are **equivalent**:
   (a) $\exists_{r>0}\ S_r(x) \subseteq U$
   (b) U is a neighborhood of x
   (c) U contains a neighborhood of x.
6. A set $U \subseteq M$ is **open** when property 5(a)(b)(c) is true for every x in U.
7. Note: a neighborhood of x is **not the same** as an open set, because if we want to check that U is an open set then we need to check property 5(a) for *every* element of U. Whereas to check if U is a neighborhood of x, we only have to check property 5(a) for one element (namely x).
8. The sets $\emptyset$ and M are **always open** (even if M does not "look" open. To understand this, selecting M means selecting *all points* to be considered. Then all r-close points to any x in M are automatically in M).
9. An **open neighborhood** is (these conditions are equivalent):
   (a) A neighborhood of x that happens to be an open set.
   (b) An open set that happens to contain x.
10. **Any** union of open sets is always open (even infinitely many sets!).
11. The intersection of **finitely many** open sets is again open.
12. x is an **isolated point** when:
    (a) $\{x\}$ is open
    (b) There is a neighborhood of x that contains just x and no other elements.
    (c) $\exists_{r>0}\, S_r(x) = \{x\}$
    (d) A sequence $x_1, x_2, \ldots$ in M can only converge to x when there is some N such that all $x_i = x$ for all $i \geq N$. In other words, when there is some tail $x_N, x_{N+1}, \ldots$ of your sequence that equals $x, x, \ldots$.

13. x is **not isolated** when

    (a) $\{x\}$ is not open.

    (b) Every neighborhood of x will contain more elements than just x.

    (c) For every $r > 0$ the set $S_r(x)$ contains more than just x.

    (d) There exists a sequence $x_1, x_2, \ldots$ in M that converges to x but where $x_n \neq x$ for every n
    (To produce such a sequence, do the following: for every n, the set $S_{\frac{1}{n}}(x) - \{x\}$ is not empty by part (c), so we can choose some x_n in $S_{\frac{1}{n}}(x) - \{x\}$. Then $x_n \neq x$ but $D(x_n, x) < \frac{1}{n}$ and therefore $x_1, x_2, \ldots$ converges to x.)

14. Let $x_1, x_2, \ldots$ be a sequence. A **tail** is what you get when you throw away the first ... (finitely many) elements. So a tail is a subsequence of the form $x_N, x_{N+1}, \ldots$ for some N (here we threw away the first $N-1$ elements).

15. $x_1, x_2, \ldots$ **converges** to x when

    (a) For every $\epsilon > 0$ the sequence has a tail contained in $S_\epsilon(x)$.

    (b) $\forall_{\epsilon>0}\, \exists_N\, \forall_{i \geq N}\, D(x_i, x) < \epsilon$

    When these equivalent properties hold then we say that x is the limit of the sequence $x_1, x_2, \ldots$.
    The most boring convergent sequences are those that have a tail that is constant. Such a sequence obviously converges. If x is isolated, then item 12(d) says that only boring sequences can converge to x.
    However, if x is not isolated, then there are more interesting sequences that converge to x, see item 13(d).

16. M is **discrete** when

    (a) Every x in M is isolated.

    (b) $\{x\}$ is open for every $x \in M$.

    (c) Every set $U \subseteq M$ is open.

17. A set $F \subseteq M$ is **closed** when

    (a) If a sequence $x_1, x_2, \ldots$ in F converges to x then x must be in F.

    (b) If $S_r(x) \bigcap F$ is not empty for every $r > 0$ then $x \in F$.

    (c) If $F \bigcap U \neq \emptyset$ for every neighborhood U of x then $x \in F$.

    (d) If every neighborhood of x intersects F (if every neighborhood of x has element(s) in common with F) then $x \in F$.

    (e) The complement of F is open, i.e. $F^c = M - F$ is open.

    (f) F contains all of its limit points (x is a limit point of $F \Longrightarrow x \in F$).

18. A point x is called a **limit point** of A if there is a sequence in $A - \{x\}$ that converges to x.

19. $\overline{A}$ is called the **closure** of the set A.

    (a) $\overline{A}$ is the union of A and all of its limit points.

    (b) $\overline{A}$ is the smallest closed set that contains A.

    (c) $\overline{A}$ is the intersection of all closed sets that contain A.

    (d) $x \in \overline{A} \iff$ every neighborhood of x intersects A.

    (e) $x \in \overline{A} \iff \exists$ a sequence $x_1, x_2, \ldots \in A$ that converges to x.

    (f) $x \in \overline{A} \iff \forall_{\epsilon > 0}$ there is a point in A that is ϵ-close to x.

20. x is a **limit point** of A if x is in the closure of $A - \{x\}$.

21. If $x_1, x_2, \ldots$ converges to x and $y_1, y_2, \ldots$ converges to y, then $D(x_1, y_1), D(x_2, y_2), \ldots$ converges to $D(x, y)$.

22. The diameter of a set A is the supremum of $\{D(x, y) | x, y \in A\}$.

23. If A is a set, then the diameter of A equals the diameter of $\overline{A}$. To prove this, you need item 21.

24. The union of *finitely many* closed sets is again closed.

25. The intersection of closed sets (even if you take infinitely many closed sets!) is again closed.