Answers Test 3, Intro Advanced Math. April 15, 2019.

1. Let M be a metric space and $x \in M$. Write down the negation of the following statement. In words, what does your answer say?

$$\forall_{r>0} \exists_{p \in M} \quad p \neq x \wedge D(p,x) < r$$

   Answer:

$$\exists_{r>0} \forall_{p \in M} \quad p = x \vee D(p,x) \geq r$$

   That's what items 12(b) and 12(c) say, so x is an **isolated point**.

2. For each of the following sets in the metric space $M = \mathbb{R}$, mention if it is open, closed, both, or neither. For each set A that is not closed, write down its closure $\overline{A}$:

   $\emptyset$: **both**
   $[0, \infty)$: **closed**
   $\mathbb{Q} - \{0\}$: **neither**. The closure is $\mathbb{R}$.
   $\mathbb{R} - \{0\}$: **open**. The closure is $\mathbb{R}$.
   $(0, 1]$: **neither**. The closure is $[0, 1]$.
   $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$: **neither**. The closure has 1 more point, namely 0.

3. Let M be a metric space, A a closed subset, and p an isolated point. Prove that $A - \{p\}$ is closed.

   Proof#1: $\{p\}$ is open because p is isolated. Then its complement $M - \{p\}$ is closed. Then $A - \{p\} = A \bigcap (M - \{p\})$ is an intersection of closed sets and thus closed.

   Proof#2: The complement of $A - \{p\}$ is $A^c \bigcup \{p\}$ which is the union of two open sets, and thus open. Then $A - \{p\}$ is closed.

   Proof#3: Let $x_1, x_2, \ldots$ be a sequence in $A - \{p\}$ and suppose it converges to x. To prove: $x \in A - \{p\}$. Now $x \in A$ because A is closed and the sequence is in A. Remains to prove: $x \neq p$. By contradiction: If $x = p$ then 12(d) says that $p, p, \ldots$ is a tail of our sequence which contradicts the fact that the sequence is in $A - \{p\}$.

   Proof#4: If A is closed and $A - \{p\}$ is not closed, then we can draw two conclusions: (1) $p \in A$, and (2) the smallest closed set that contains $A - \{p\}$ must be A, in other words $A = \overline{A - \{p\}}$ (see item 19(b)). Then p is a limit point of A (item 20) which implies (see items 18 and 13(d)) that p is not isolated, contradicting an assumption.

4. Let M be a metric space, A is a subset of M, and $x \in M$. Suppose that $x \in \overline{A}$ and $x \notin A$. Show that x is not isolated.

   Proof#1: If $x \in \overline{A}$ and $x \notin A$ then x is a limit point of A (see item 20) which implies (see items 18 and 13(d)) that x is not isolated.

   Proof#2: By 19(e) if $x \in \overline{A}$ then there is a sequence in A that converges to x. But if $x \notin A$ then every member of that sequence is $\neq x$ so then x is not isolated by 13(d).

   Proof#3: The set $\overline{A}$ is closed, so if x were isolated, then by Exercise 3 we would see that $F := \overline{A} - \{x\}$ is still closed. Then F is a closed set that contains A (use $x \notin A$). But then $\overline{A}$ is not the smallest closed set that contains A since $x \in \overline{A}$ and $x \notin F$. That contradicts 19(b).

5. Let M be a metric space, and let A be a subset of M. For $x \in M$ we say that the distance from x to A is less than 1 if there exists some $a \in A$ with $D(a, x) < 1$. Let U be the set of points in M that have distance less than 1 to A. In other words

   $$U = \{x \in M \mid \exists_{a \in A} \;\; D(a, x) < 1\}$$

   Show that U is an open subset of M.

   Proof#1:

   $$U = \{x \in M \mid \exists_{a \in A} \;\; x \in S_1(a)\} = \bigcup_{a \in A} S_1(a) = \bigcup \text{ open sets} = \text{open}$$

   Proof#2: Let $x \in U$. To prove: any of the items 5(a), 5(b), or 5(c). Now $x \in S_1(a)$ for some $a \in A$. But $S_1(a)$ is open and contains x, so $S_1(a)$ is an open neighborhood of x (see item 9(b)), so we proved 5(c).

   Proof#3: Let $x \in U$. Then $D(a, x) < 1$ for some $a \in A$. To prove: $\exists_{r>0} \;\; S_r(x) \subseteq U$. Proof: take $r = 1 - D(a, x)$.

   Remains to prove: $S_r(x) \subseteq U$ (if you omitted this you'll still get full credit. You can prove $S_r(x) \subseteq S_1(a) \subseteq U$ with the triangle inequality).

   Proof#4: Lets prove that U^c is closed with item 17(a). Take a sequence $x_1, x_2, \ldots$ in U^c that converges to x. To prove $x \in U^c$. A point is in U^c if it has distance ≥ 1 from any a in A. So for any $a \in A$ we have $D(x_1, a) \geq 1$, $D(x_2, a) \geq 1$, etc. The last theorem in section 4.3 says that if $x_1, x_2, \ldots$ converges to x then $D(x, a)$ is the limit of $D(x_1, a), D(x_2, a), \ldots$. All of those are ≥ 1 so the limit $D(x, a)$ must be ≥ 1 as well. So $D(x, a) \geq 1$, for arbitrary $a \in A$, and hence $x \in U^c$.