Why I do what I do

An informal research statement

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When I first encountered a computer algebra system, the fact that the computer could integrate functions better than I could was like magic. I wanted to know how that worked. I took a course in computer algebra from professor Levelt and learned how the Risch algorithm works. In the algebraic case this involves an interesting problem from algebraic geometry (bounding the torsion of a divisor in the Jacobian using reduction at good primes). That the problem of integration involves problems from pure math was fascinating to me. My mind was made up: I wanted to work in computer algebra, so I asked professor Levelt to be my advisor.

Besides integrating functions, what do most people use computer algebra for? The answer I heard most often was: differential equations. This too, I learned from professor Levelt, involved interesting mathematics. For instance, he told me that, in the same way as Galois theory dictates if and how polynomials can be solved in terms of radicals, there was an analogue Galois theory for linear differential equations. Well, that sounded extremely interesting. I thought that Galois theory was just for polynomials, and now all of a sudden the same ideas work for an entirely different problem! I had to know more about that. As it turned out, though I didn't know it at the time, differential equations involve a huge number of pure math problems, many more than the Risch algorithm. So this was going to be a topic that was not only very useful to people, it was going to be very interesting mathematically as well. This would be a main area for my research.

I do at times work on other topics as well. For instance, I needed to compute gcd's of polynomials, with algebraic coefficients, and this was too slow, unnecessarily slow it seemed to me. So I worked together with Mike Monagan to address this issue. And a combinatorial problem in solving difference equations, the way I wanted to deal with that turned out to be applicable to another problem, factoring polynomials, so I wrote a paper on that too.

As a graduate student, I worked on two topics, algebraic curves and differential equations. The advantage of that was that whenever I got stuck in one, I could work on the other for a while. The reason I got into algebraic curves is because professor Levelt asked me if I could write an integral basis algorithm for algebraic function fields, and because of what I learned from professor Steenbrink. Professor Levelt was interested in such an algorithm because it is an ingredient of the Risch algorithm in the algebraic case (he asked me to try this when I told him that my method for understanding the number theory course I was taking at the time was by implementing those things on the computer, one of which was computing an integral basis in a number field).

But the magic that first got me into computer algebra still calls. You start with an equation and see no way to solve it by hand. Take for example this one:

$$y'''' + \frac{3-2x^2}{8x^2}y'' + \frac{3}{4x^3}y' - \frac{567}{256x^4}y = 0.$$

Can you find an exact formula for the solutions?

My goal is this: That to solve this equation, all you would have to do is click on a button, and out come the solutions. One of the solutions is

$$x^2\left(I_0(\frac{x}{4})I_{1/4}(\frac{x}{4}) - I_1(\frac{x}{4})I_{5/4}(\frac{x}{4})\right)$$

where I_ν is the Bessel I function with parameter ν. The other solutions look like this as well. Even if you know how these algorithms work, even if you write them, it's still fascinating really that this works.