1. In this problem, we will develop a stage-based model for loggerhead sea-turtles. As discussed in class, the stages are: hatchlings ($h$); small juveniles ($sj$); large juveniles ($lj$); subadults ($sa$); and adults ($a$). The numbers of turtles in each class are counted once every year. The following life history model depicts how the turtles progress through life.

(i) The Leftkovitch Matrix below describes how the turtles progress through the various stages of life:

$$
\begin{bmatrix}
0 & 0 & 0 & 5.45 & 69.39 \\
0.67 & 0.70 & 0 & 0 & 0 \\
0 & 0.05 & 0.77 & 0 & 0 \\
0 & 0 & 0.02 & 0.76 & 0 \\
0 & 0 & 0 & 0.06 & 0.88
\end{bmatrix}
$$
The non-zero entries in the above matrix correspond to fecundities and the probabilities that individuals will either survive and remain in the same class in 1 year, or will progress to the next stage of life. In the life history schematic above, fill in these numerical probabilities next to each arrow.

(ii) For an initial population size of 30,000 hatchlings, 51,000 small juveniles, 17,000 large juveniles, 1500 subadults and 500 adults, graph how the population of turtles progresses with time. On a second graph, also plot the age distribution (really stage distribution) versus time, that is, plots of the ratio of the population in each stage at time \( n \) to the total population at time \( n \). Does a stable stage distribution emerge?

2. For higher organisms, the growth rate is more complicated than the logistic growth model. For example, reproduction could be reduced if it becomes too difficult to find a mate. An alternate higher order model to the logistic growth model is one modeling the Allee effect, that is, if too few individuals cannot sustain the population. A differential equation for this model is given by:

\[
\frac{dN}{dt} = \lambda N \left(1 - \frac{N}{K}\right) (N - a).
\]

Here \( \lambda, K \) and \( a \) are (positive) model parameters with \( a < K \).

(i) Find all the fixed points of the system and classify their stability. (You may restrict yourself to nonnegative values of \( N \)).

(ii) Sketch various solutions \( N(t) \) for a range of initial conditions - \( N(0) < a/2, a/2 < N(0) < a, a < N(0) < K/2, K/2 < N(0) < K \) and \( N(0) > K \).

(iii) Discuss how the above differs qualitatively from solutions to the logistic equation.

3. For the following system, draw null clines in the \( xy \) phase plane, identify steady states, and indicate the direction of flow in each region of the plane. Sketch a few plausible trajectories. Also calculate the Jacobian and classify each stay state as stable, unstable or non-hyperbolic, and, if appropriate, as node, spiral or saddle point. If saddle point, indicate the stable manifold in your phase portrait.

\[
\dot{x} = x - x^3, \quad \dot{y} = -y
\]
4. Consider the following Lotka-Volterra model of two competing species:

\[
\frac{dx}{dt} = x(1 - x) - Axy,
\]

\[
\frac{dy}{dt} = y(1 - y) - Bxy.
\]

Here: \(x(t)\) and \(y(t)\) represent population densities (number per unit area) of the two interacting species; \(t\) is time; and \(A\) and \(B\) are positive, constant model parameters.

(a) Find all possible steady states and investigate their local linear stability by computing the Jacobian matrix. You will need to consider the following four cases: \(A, B > 1; A > 1, B < 1; A < 1, B > 1;\) and \(A, B < 1\).

(b) Depict your findings in in phase portrait diagrams.

(c) From your findings, you should be able to see that the model demonstrates: (i) the Principle of Competitive Exclusion where one species always outcompetes the other; (ii) the Founder Effect, where either species could dominate, and the outcome is dependent on initial conditions; and (iii) co-existence, where both species can co-exists. Indicate which of the cases from (a) correspond to each of these scenarios.

5. Consider a lake with some fish attractive to fisherman. We wish to model the fish-fisherman interaction under the following assumptions:

- the fish population grows logistically in the absence of fishing;
- the presence of fishermen depresses the fish growth rate at a rate jointly proportional to the size of the fish and fisherman populations;
- fishermen are attracted to the lake at a rate directly proportional to the number of fish in the lake;
- fishermen are discouraged from the lake at a rate directly proportional to the number of fishermen already there.

(a) Write down a mathematical model for this situation, clearly defining your terms.

(b) Show that a non-dimensionalized version of the model is:

\[
\frac{du}{dt} = ru(1 - u) - uv,
\]

\[
\frac{dv}{dt} = \beta u - v,
\]
where $u$ and $v$ represent the non-dimensionalized fish and fishermen populations, respectively.

(c) Calculate the steady states of the system and determine their stability.

(d) Draw the phase plane, including the nullclines and vector field.

(e) What would be the effect of adding fish to the lake at a constant rate?

6. The following equations represent a SIS-type model of a sexually transmitted disease. The model includes births and vertical transmission, that is, children born to infected mothers are infected:

$$\frac{dS}{dt} = bS \left( 1 - \frac{N}{K} \right) - \beta SI + \gamma I,$$

$$\frac{dI}{dt} = bI \left( 1 - \frac{N}{K} \right) + \beta SI - \gamma I.$$

Here, the total population $N = S + I$. Find the basic reproduction number $R_0$ of this model (that is, a bifurcation parameter such that if $R_0 > 1$, the disease free equilibrium is unstable, and vice versa).