THE GAUSS-GREEN THEOREM FOR FRACTAL BOUNDARIES

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§1: Introduction

The Gauss-Green formula

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega, \tag{1}$$

where Ω is a compact smooth n-manifold with boundary in R^n and ω is a smooth $(n-1)$-form in R^n, is a classical part of the calculus of several variables (e.g. [Sp]).

When Ω is permitted to have positive codimension, (1) is often called Stokes' Theorem; we use "Gauss-Green" to refer to the case where Ω has codimension zero. (Note that the Gauss-Green formula is often written in the equivalent form

$$\int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, ds = \int_{\Omega} \operatorname{div} \mathbf{v}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, ds is the element of area on $\partial\Omega$, and $\mathbf{v}$ is the 1-vectorfield "dual" to ω: if $\omega = \sum(-1)^{i+1} f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$, then $\mathbf{v} = (f_1, \ldots, f_n)$.)

There has been considerable effort in the literature (e.g. [JK], [M], [P]) to extend this formula to permit integrands of less regularity by generalizing the Lebesgue integral. On the other hand, invariably the situations in which (1) holds require fairly strong hypotheses on the boundary $\partial\Omega$, e.g. that it should have sigmafinite $(n-1)$-measure, or that the gradient of the characteristic function of Ω be a vector valued measure with finite total variation [F], [P].

However there is a natural way to expand the validity of (1) to much more general boundaries while still using the ordinary Lebesgue integral; this is the topic of the present paper.

For the case of Lipschitz forms, the results of this paper follow readily from Whitney's theory of flat chains [W2]. However his approach to the Gauss-Green theorem is not widely appreciated because he focused on chains and cochains, where effectively (1) is used to define the exterior derivative. In [HN] we extend Whitney's method to treat the more general Hölder case. (Only the case $n = 2$ is discussed

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there, but the ideas go through with little change to treat closed codimension 1 submanifolds in R^n.) But that paper also uses (1) partly as a definition.

The purpose of this paper is to present (1) as a *theorem* (see Theorem B), in which the boundary is not required to be rectifiable, and the LHS is defined in a way logically independent of the RHS. We also give statements in better generality than appear in [HN]. Furthermore, since we treat only codimension 1 boundaries, the discussion is simplified, and we make every effort to arrive at our goal as economically and accessibly as possible.

The method of integrating a form over a fractal boundary in [HN] is to integrate over a smooth or PL approximation of the boundary, and then take a limit. For a variety of reasons this is not as simple as it might at first seem (see examples 1 and 2 below). For example, how is one to take the limit? This question was answered by Whitney [W2] with his "flat norm" on the space of polyhedral chains, which has the property that two chains close in flat norm will have integrals that are close, provided the integrands (forms) are properly bounded ("flat"). Here (as in [HN]) we present an extension of the flat norm which permits us to reveal examples 1 and 2 below as sharp counterexamples for the theory.

The elementary idea of simply taking limits of integrals of PL approximations must fail in complete generality, as the following classical example shows.

Example 1. [W1],[N2]. *There exists a C^1 function f and a continuously embedded arc γ such that $df = 0$ at each point of γ, but f is not constant along γ; we can choose f to be increasing along γ, $f(\gamma(0)) = 0$ and $f(\gamma(1)) = 1$.*

In this paper we ask that a good theory of geometric integration have both of the following properties:

Property A: $\omega|_\gamma = 0$ implies $\int_\gamma \omega = 0$, i.e. the value of the integral should depend only on the values of the form on the submanifold over which it is being integrated, and

Property B: formula (1) where Ω is a topological submanifold with boundary, of arbitrary codimension.

Example 1 (with $\gamma = \Omega$) shows that we cannot have both A and B for all continuous forms and all continuous submanifolds. (See also example 3 in section 4 below.) However such counterexamples cannot be very smooth: the $C^{1,1}$ Morse-Sard theorem [B] implies that f cannot be $C^{1,1}$, and in fact it is shown in [N1] that f can be at most of class C^s, where s is the Hausdorff dimension of the curve γ. Hence it is reasonable to expect that properties A and B should be attainable if we assume some minimum regularity of the form relative to the dimension of the submanifold or its boundary.

In the context of Whitney's flat theory, γ of example 1 represents a flat chain and df, since it is exact, represents a flat cochain; yet Property A fails. To recover Property A we must require the form to be Lipschitz as well (and any Lipschitz form represents a flat cochain) —see Theorem A'. Similarly, we also obtain Property A for Hölder forms with an appropriate condition on the dimension of the boundary —see Theorem A.

In this paper we restrict attention to submanifolds Ω of codimension zero. The case of arcs in the plane is treated in [HN]; for the more general Stokes' theorem see [H].

It will turn out that the essential hypothesis for integration is roughly that the *box* dimension of the boundary should not be too large when compared with the Hölder class of the form to be integrated (see Theorems A and B below).

Even when all curves in the discussion are PL, it is still possible for the limit of the integrals to disagree with the integral of the limit, as the following example shows:

Example 2. [HN]. *Let σ denote a compact line segment in the plane. There exists a sequence σ_n of PL embedded curves coterminal with σ and a Hölder form ω so that*

(i) σ_n tends to σ in the Hausdorff metric on compact subsets of the plane, and
(ii) $\int_\sigma \omega = 0$, but $\int_{\sigma_n} \omega > 1$ for all n.

This means we must be careful in specifying the proper topology on curves or chains; we do this by means of the "d-flat norm" below. Example 2 also illustrates the peculiar difficulty of Hölder forms in the theory: for smooth forms the integrals will converge properly if σ_n and σ enclose an area tending to zero. (This is convergence in Whitney's "flat norm".) For forms with less regularity, the d-flat norm is needed.

In sections 2 and 3 we define, in a geometric way, the integral of a *d-flat* $(n-1)$-form over a *d-summable* boundary, for $n-1 < d \leq n$. (Any $(d-n+1)$-Hölder $(n-1)$-form is d-flat.) Then in section 4 we prove Property A for $(d-n+1)$-Hölder forms and Property B for all d-flat forms satisfying a Sobolev condition.

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§2: Abstract Theory; flat chains and the definition of the integral.

There is a standard theory of polyhedral chains in R^n, which we summarize here. (See, e.g., [W2, Ch 5] for more details.)

An *affine n-simplex* in R^n is the oriented convex hull $< p_0, ..., p_n >$ of $n+1$ affinely independent points. If $\sigma_1, ..., \sigma_m$ are non-overlapping oriented affine n-simplices, and $a_1, ..., a_m$ are real coefficients, the expression $A = \Sigma a_i \sigma_i$ determines a *polyhedral n-chain* in R^n.

We want such a chain to be independent of choice of subdivision, so we employ the following device. Define a function $A(p)$ to have value a_i or $-a_i$ for p in $\mathrm{int}(\sigma_i)$, according to whether σ_i is oriented positively or negatively relative to the standard orientation on R^n, and zero for p outside the interiors of the simplices σ_i.

Then identify two polyhedral chains A and B if $A(p) = B(p)$ except perhaps on a finite number of simplices of dimension less than n. This space of equivalence classes forms a real vector space in a natural way, denoted P_n. The elements of P_n are the polyhedral n-chains in R^n.

For example, if $A = \Sigma a_i \sigma_i$ and $B = \Sigma b_i \tau_i$, then $tA = \Sigma t a_i \sigma_i$ and $A + B = \Sigma(a'_i + b'_i)\mu_i$, where $\{\mu_i\}$ is a common subdivision of A and B and $A = \Sigma a'_i \mu_i, B = \Sigma b'_i \mu_i$.

For $m < n$, a polyhedral m-chain in R^n is a finite set of oriented m-planes in R^n, together with a polyhedral m-chain in each. The space of polyhedral m-chains in R^n is denoted P_m.

The *boundary* of an n-simplex $\sigma =< p_0, ..., p_n >$ is defined to be the polyhedral $(n-1)$-chain

$$\partial\sigma = \Sigma(-1)^i < p_0, ..., \hat{p}_i, ..., p_n > .$$

Given an m-simplex $\sigma =< p_0, ..., p_m >$ in R^n and a continuous m-form ω, we have the standard Riemann integral $\int_\sigma \omega$. This is defined, for example, by letting τ_m be the standard oriented m-simplex in R^m, $\tau_m =< 0, e_1, e_2, ..., e_m >$, choosing an affine parametrization $\phi : \tau_m \to \sigma$ so that $\phi(e_i) = p_i, i = 1, ..., m$, and letting

$$\int_\sigma \omega = \int_{\tau_m} \omega(\phi(s))(D\phi(s)e_1, ..., D\phi(s)e_m)ds_1 ds_2 ... ds_m. \tag{2}$$

For a polyhedral m-chain $A = \Sigma c_i \sigma_i$, then, we let

$$\int_A \omega = \Sigma c_i \int_{\sigma_i} \omega$$

for any continuous m-form ω. We can relax the requirement of continuity if we wish, so long as the integrand in the RHS of (2) is integrable in some suitable sense, e.g. Lebesgue.

Definition. *An m-form ω in R^n is* m-**measurable** *if for each affine m-simplex σ in R^n with affine parametrization $\phi : \tau_m \to \sigma$, the function*

$$\omega(\phi(s))(D\phi(s)e_1, ..., D\phi(s)e_m)$$

is measurable on τ_m.

Example. An m-form in R^n which is H^m-a.e. equal to a continuous m-form is m-measurable. (H^m is Hausdorff m-measure).

d-mass.

For $0 < d \leq m$, the d-mass $M_d(A)$ of a polyhedral m-chain in R^n is

$$M_d(A) = \inf\{\Sigma|a_i||\sigma_i|^d : A = \Sigma a_i \sigma_i\}.$$

(Here $|\sigma|$ denotes the diameter of σ as a subset of R^n.)

It is straightforward to check that M_d is a norm on P_m ([HN]). Note that M_m is equivalent to Whitney's *mass* [W2].

For $d = n$, the d-mass is comparable to the volume, but when $d < n$ the d-mass is difficult to compute in specific cases. For example, if σ is an equilateral triangle in the plane, then $M_d(\sigma) = |\sigma|^d$, but if σ is very long and thin, then $M_d(\sigma) << |\sigma|^d$.

The sums in the definition of M_d are reminiscent of those in the definition of Hausdorff d-measure. However, we do not, as in that case, take the limit of finer and finer subdivisions, since for $d < m$ this would diverge to infinity for chains with nontrivial support. The infimum in the definition of M_d occurs instead when the sub-simplices are in some sense as large as possible. This is what makes computation of M_d a subtle matter.

d-flat norm.

For $n - 1 < d \leq n$, we define the d-flat norm $|A|_d$ of the polyhedral $(n-1)$-chain A as follows:

$$|A|_d = \inf\{M_{n-1}(S) + M_d(T) : A = S + \partial T\}.$$

It is easy to check that this defines a seminorm. To see that it is a norm, given $A \neq 0$, choose a C^∞ $(n-1)$-form ω so that $\int_A \omega \neq 0$. Then $|A|_d \neq 0$ is a consequence of the following temporary lemma (which will be shortly supplanted by (3)):

Lemma 1. *If $A \in P_{n-1}$ is supported in a disk of radius $r \geq 1$, and ω is a smooth $(n-1)$-form, then*

$$|\int_A \omega| \leq r|\omega|_{C^1}|A|_d.$$

Proof. If τ is an n-simplex, then volume$(\tau) \leq r|\tau|^d$. It is then easy to check that

$$|\int_{\partial T} \omega| = |\int_T d\omega| \leq rM_d(T)|dw|_0$$

for any polyhedral n-chain T.

Also for any $(n-1)$-chain S in P_{n-1}, $|\int_S \omega| \leq |\omega|_0 M_{n-1}(S)$.

Thus if $A = S + \partial T$,

$$|\int_A \omega| \leq r|\omega|_{C^1}(M_{n-1}(S) + M_d(T)),$$

and this completes the proof.

We now have the linear spaces P_{n-1} and P_n, equipped with norms $|.|_d, M_d$, respectively. Define $\mathcal{E}_d$ to be the completion of (P_n, M_d) and $\mathcal{C}_d$ to be the completion of $(P_{n-1}, |.|_d)$. The boundary operator $\partial : P_n \to P_{n-1}$ satisfies $|\partial A|_d \leq M_d(A)$ and therefore extends to a unique bounded linear operator

$$\partial : \mathcal{E}_d \to \mathcal{C}_d$$

satisfying the same inequality.

We remark that when $d' < d$, $\mathcal{C}_{d'}$ embeds continuously into $\mathcal{C}_d$, since the d'-flat topology is finer than the d-flat topology. This becomes clearer if we make the optional stipulation in the definition of M_d that all subdividing simplices are to have diameter at most one. In this case, we obtain the simple inequalities

$$M_d \leq M_{d'} \text{ and } |.|_d \leq |.|_{d'} \text{ whenever } d' \leq d.$$

d-flat $(n-1)$-forms.

Let ω be an $(n-1)$-measurable $(n-1)$-form on R^n. (For simplicity, assume henceforth that ω has compact support.)

Definitions.

$$|\omega| = \inf\{C : |\int_\sigma \omega| \leq CM_{n-1}(\sigma), \textit{for all } (n-1)\textit{-simplices } \sigma\},$$

$$||\omega|| = \inf\{C : |\int_{\partial\tau} \omega| \leq CM_d(\tau), \textit{for all } n\textit{-simplices } \tau\},$$

and

$$|\omega|^d = \inf\{C : |\int_A \omega| \leq C|A|_d, \ \textit{for all } A \in P_{n-1}\}.$$

Note that $|.|^d$ is the dual norm to $|.|_d$ with respect to the linear pairing $\omega.A = \int_A \omega$.

The following proposition helps in understanding the norm $|.|^d$. The proof is straightforward, but is omitted since we will not use this proposition here.

Proposition. [HN]. $|\omega|^d = \max(|\omega|, ||\omega||)$.

We denote by F^d the space of $(n-1)$-measurable $(n-1)$-forms ω in R^n for which $|\omega|^d < \infty$. This is the space of d-**flat** $(n-1)$**-forms**.

Note that our definitions imply

$$|\int_A \omega| \leq |A|_d |\omega|^d \tag{3}$$

for all $A \in P_{n-1}, \omega \in F^d$.

If $d' \leq d$, then since $|.|^d$ is dual to $|.|_d$, we get $||^{d'} \leq ||^d$. Hence $F^{d'}$ embeds naturally into F^d.

Extension of the integral to the full space of d-flat forms and chains.

So far we have been discussing the standard Lebesgue integral for forms on polyhedral chains. But by virtue of (3), the bilinear operator

$$\int : P_{n-1} \times F^d \to R$$

extends uniquely to $\mathcal{C}_d \times F^d$, satisfying the same inequality. We will denote this extended operator by the symbol $\int^\flat$. Thus $|\int_A^\flat \omega| \leq |A|_d |\omega|^d$ for all $A \in \mathcal{C}_d, \omega \in F^d$.

Remarks.

1. This definition of $\int^\flat : \mathcal{C}_d \times F^d \to R$ is equivalent to defining, for $A \in \mathcal{C}_d$, $\int_A^\flat \omega = \lim \int_{A_k} \omega$, where $\{A_k\}$ is any sequence of polyhedral chains tending to A in the $|.|_d$-topology. That the limit exists and is independent of the sequence A_k is a simple consequence of (3).

2. We could consider the larger space $\mathcal{C}^d = (\mathcal{C}_d)^*$ of d-flat $(n-1)$-cochains and get a satisfactory theory [HN]. However we will restrict our attention to forms in this paper for the sake of their more concrete geometric meaning.

3. If ω is a $(d-n+1)$-Hölder $(n-1)$-form, then $\omega \in F^d$. (This is a straightforward exercise.) In fact

$$|\omega|^d \leq C|\omega|_{d-n+1}$$

for some constant C, where the norm on the right is the $(d-n+1)$- Hölder norm.

Hence for such forms,

$$|\int_A^\flat \omega| \leq C|\omega|_{d-n+1}|A|_d.$$

The reader may wish to think of such Hölder forms rather than the more general d-flat forms.

4. $\mathcal{C}_n$ is simply the original space of flat $(n-1)$-chains as defined in [W2].

§3: The geometric theory

So far we have defined integration on a large space $\mathcal{C}_d$ of abstract chains. Given a geometric boundary $\partial\Omega$, we wish to make use of this theory by identifying $\partial\Omega$ with a unique element $(\partial\Omega)^\flat \in \mathcal{C}_d$ –and then we will simply define $\int_{\partial\Omega}\omega$ to be $\int^\flat_{(\partial\Omega)^\flat}\omega$.

Our method is to identify Ω with an element $\Omega^\flat$ of $\mathcal{E}_d$, and then $(\partial\Omega)^\flat$ will be $\partial(\Omega)^\flat$.

Definition. *A* **Jordan domain** *Ω in R^n is a bounded oriented connected open subset of R^n whose boundary is a compact topological hypersurface.*

(For $n = 2$, this yields the usual notion of Jordan domain in the plane.)

Definition. *An* **affine decomposition** *of an open set Ω is a collection $\mathcal{T}$ of non-overlapping affine n-simplices whose interiors are all contained in Ω and such that for some set E of Lebesgue measure zero, $\cup\mathcal{T} \supset \Omega\backslash E$. If $E = \emptyset, \mathcal{T}$ is called* **proper**.

The d**-sum** *of any decomposition $\mathcal{T}$ is the (possibly infinite) quantity $\sum_{\tau\in\mathcal{T}} |\tau|^d$.*

The principal example of a proper decomposition is the *Whitney decomposition* $\mathcal{W}$ of Ω by binary cubes, defined as follows (e.g. [S]):

A cube Q is called a k-cube if it is of the form

$$[l_1 2^{-k}, (l_1 + 1)2^{-k}] \times ... \times [l_n 2^{-k}, (l_n + 1)2^{-k}]$$

where $k, l_1, ..., l_n$ are integers.

When Ω is bounded, there is a smallest k_0 such that some k_0-cube and all its neighbors are contained in Ω. Then we can inductively define $\mathcal{W}^k$ to be the collection of all k-cubes $Q \subset \Omega$ satisfying

(a) every k-cube touching Q is contained in Ω, and

(b) Q is not contained in any cube in $\mathcal{W}^j$ for $j < k$.

Let $\mathcal{W} = \cup_{k_0}^\infty \mathcal{W}^k$.

We will use the Whitney decomposition of Ω to define an approximating sequence of polyhedral n-chains as follows. Any cube determines a polyhedral n-chain in the obvious way, so we can let

$$W_k = \Sigma\{\tau \in \mathcal{W} : \tau \in \mathcal{W}^j \text{ for some } j \leq k\} \in P_n.$$

We then define $\Omega^\flat \equiv \lim W_k \in \mathcal{E}_d$. Our job now is to show that this limit exists under appropriate geometric assumptions on Ω. To do this, we must first introduce a geometric notion of summability.

Definitions. Given a bounded set $X \subset R^n$, let $N_X(\epsilon)$ be the number of ϵ-balls needed to cover X. Then the **box dimension** of X is defined by

$$\dim X = \limsup_{\epsilon\to 0} \frac{\log N_X(\epsilon)}{-\log\epsilon}.$$

Note that in defining N_X if we permitted ourselves only the use of k-cubes for $2^{-k} \leq \epsilon < 2^{-k+1}$, this would change N_X by at most a bounded factor, and hence $\dim X$ not at all. Therefore we are free to think of N_X this way if we wish. In either case, N_X is a monotone function of ϵ.

We can think of $\dim X$ as measuring the asymptotic maximum exponential rate of increase of $N_X(\epsilon)$ as $\epsilon \to 0$. (This number agrees with the topological dimension for smooth submanifolds, and agrees with the Hausdorff dimension for self-similar sets.) However the number $\dim X$ is slightly too crude for our purposes.

Definition. A bounded set $X \subset R^n$ is d**-summable** if the improper integral $\int_0^1 N_X(x)x^{d-1}dx$ converges.

Note. If $\dim X < d$, then $N_X(\epsilon) \leq O(\epsilon^{-d'})$, for $\dim X < d' < d$, and hence X is d-summable.

If $\dim X = d$, then X is still d-summable if N_X does not grow too quickly, e.g. if $N_X(\epsilon)\epsilon^d(\log \epsilon)(\log\log\epsilon)^2$ remains bounded as ϵ tends to zero.

The importance of this definition lies in

Lemma 2. *If Ω is a bounded open subset of R^n and $\partial\Omega$ is d-summable, then the d-sum $\sum_{Q\in\mathcal{W}} |Q|^d$ of the Whitney decomposition $\mathcal{W}$ of Ω is finite.*

Proof. Write $N = N_{\partial\Omega}$.

The main observation is that the number of k-cubes in $\mathcal{W}$ is at most a constant (depending only on n) times $N(2^{-k})$. This is because, by virtue of the definition of $\mathcal{W}$, each k-cube Q of $\mathcal{W}$ is contained in a $(k-1)$-cube Q' that touches a $(k-1)$-cube Q'' meeting $\partial\Omega$. (Otherwise Q' would be in $\mathcal{W}$ instead of Q.) The number of such cubes Q'' is controlled by $N(2^{-k+1}) \leq 2^n N(2^{-k})$, and hence the number of such Q is controlled by a constant times $N(2^{-k})$.

The d-sum of the k-cubes of $\mathcal{W}$ is therefore less than $C(n)N(2^{-k})2^{-kd}$, where $C(n)$ is some constant depending only on n.

This means that the d-sum of $\mathcal{W}$ is finite if

$$\sum_{k=k'}^{\infty} N(2^{-k})2^{-kd} < \infty$$

where $2^{-k'}$ is the size of the largest cube in $\mathcal{W}$.

This is true iff $\int_0^\infty N(2^{-y})2^{-dy}dy < \infty$, and this, by means of the change of variable $x = 2^{-y}$, is our hypothesis.

Now we can state

Proposition 1. *If Ω is a Jordan domain in R^n and $\partial\Omega$ is d-summable, then $\Omega^\flat \equiv \lim \mathcal{W}_k$ exists in $\mathcal{E}_d$.*

Proof. By Lemma 2, $\sum_{Q\in\mathcal{W}} |Q|^d < \infty$. Hence, if $k < j$,

$$M_d(W_k - W_j) \leq \sum_{i=k+1}^{j} \sum_{Q\in\mathcal{W}^i} |Q|^d \to 0 \text{ as } k \to \infty.$$

This means that the sequence $\{W_k\}$ in P_n is M_d-Cauchy, and so converges to some element of $\mathcal{E}_d$.

Definition. If Ω is any Jordan domain with d-summable boundary and $\omega \in F^d$, then we define

$$\int_{\partial\Omega} \omega \equiv \int_{\partial(\Omega^\flat)}^{\flat} \omega.$$

The meaning of this definition is that $\int_{\partial\Omega}\omega$ can be computed as $\lim \int_{\partial W_k}\omega$, or indeed $\lim \int_{A_k}\omega$ where $\{A_k\}$ is any other sequence in P_{n-1} tending to $\partial(\Omega^\flat)$.

Remark. This definition of $\Omega^\flat$ does not depend on our choice of the Whitney decomposition. Indeed, if $\{T_k\}$ is any sequence of PL Jordan domains contained in Ω and similarly oriented, then $\lim T_k = \Omega^\flat$ provided that there is, for each k, an affine decomposition $\mathcal{T}_k$ of $\Omega \setminus T_k$ such that the d-sum of $\mathcal{T}_k$ tends to zero as $k \to \infty$. (This is proved for $n = 2$ in [HN], Lemma 5.1, and the proof in the general case is similar.)

More simply, one can show directly that any *regular* decomposition can take the place of the Whitney decomposition in the above definition and yield the same element $\Omega^\flat$. (A regular decomposition is an affine decomposition $\mathcal{T}$ such that for some $C > 1$ and for each $\tau \in \mathcal{T}$,

(i) $|\tau|^n \leq C\text{vol}(\tau)$, and

(ii) $(1/C)\text{dist}(\tau, \partial\Omega) \leq |\tau| \leq C\text{dist}(\tau, \partial\Omega)$.)

For consistency we need to know that if our domain already represents an element of P_n, and hence $\int_{\partial\Omega} \omega$ is already defined, we have $\int_{\partial\Omega} \omega = \int^\flat_{\partial(\Omega^\flat)} \omega$. By (3), this follows from

Proposition 2. *Suppose Ω is the interior of a finite union of oriented nonoverlapping affine n-simplices $\sigma_1, ..., \sigma_N$. Let $\overline{\Omega} = \sum \sigma_i \in P_n$.*

Then $\Omega^\flat \equiv \lim W_k = \overline{\Omega}$.

We relegate the proof to the Appendix.

§4: The Gauss-Green Theorem

We wish to show that the extended integral we have defined satisfies properties A and B of section 1. First,

Theorem A. *Let $d \in (n-1, n]$. If ω is H^{n-1}-a.e. equal to a $(d-n+1)$-Hölder form, and Ω is a Jordan domain in R^n such that $\partial\Omega$ is d-summable,*

then $\int_{\partial\Omega} \omega = 0$ if $\omega|_{\partial\Omega} = 0$.

Proof. For purposes of integration we are free to assume that ω is actually $(d-n+1)$-Hölder.

Assume $\omega|_{\partial\Omega} = 0$. Think of ∂W_k as representing a finite union of PL $(n-1)$-manifolds, approximating $\partial\Omega$ in the Hausdorff metric. For $x \in \partial W_k, Q \in \mathcal{W}^k$ a cube containing x, and $p \in \partial\Omega$ any point minimizing the distance from x to $\partial\Omega$, we have

$$|\omega(x)| = |\omega(x) - \omega(p)| \leq |\omega|_{d-n+1}|x-p|^{d-n+1} \leq C|\omega|_{d-n+1}|Q|^{d-n+1}$$

for some constant C depending only on n.

If S denotes an $(n-1)$-face of ∂W_k and $Q \in \mathcal{W}^k$ is the k-cube containing S, we then have

$$|\int_S \omega| \leq C|\omega|_{d-n+1}|Q|^{d-n+1} \text{ vol}(S) \leq C|\omega|_{d-n+1}|Q|^d.$$

Each face of ∂W_k is one of the $2n$ faces of some $Q \in \mathcal{W}^k$. Therefore

$$|\int_{\partial W_k} \omega| \leq 2nC|\omega|_{d-n+1} \sum_{Q \in \mathcal{W}^k} |Q|^d \to 0$$

as k tends to infinity. //

The following example shows that "$(d-n+1)$-Hölder" cannot be replaced by "d-flat" (nor weakened to $(d'-n+1)$-Hölder for any $d' < d$).

Example 3. *For any $d \in (1,2]$, there is a d-flat 1-form ω in the plane, and a Jordan domain Ω with d-summable boundary, so that $\omega|_{\partial\Omega} = 0$ but $\int_{\partial\Omega} \omega \neq 0$.*

To show this, we can construct, by the same technique as in Example 1, a compact arc γ with dimension $d' \in (1,d)$ and a $C^{d'}$ function $f : R^2 \to R$ so that

$$df = 0 \text{ on } \gamma, f(\gamma(0)) = 0, \text{ and } f(\gamma(1)) = 1.$$

For convenience of this example, we can also arrange the following extra conditions:

(a) $\gamma(0) = (0,1/2)$, $\gamma(1) = (1,1/2)$, and, except for these endpoints, γ is contained in the open unit square $U = \{(x,y) : 0 < x < 1 \text{ and } 0 < y < 1\}$, and

(b) $f \equiv 0$ on $\{(x,y) : x \leq 0\}$ and $f \equiv 1$ on $\{(x,y) : x \geq 1\}$.

Now let σ be any smooth arc with $\sigma(0) = \gamma(1)$ and $\sigma(1) = \gamma(0)$, and such that σ is, except for endpoints, disjoint from $\bar{U}$. Then $\sigma \cup \gamma$ forms the boundary of a Jordan domain Ω, and this boundary is d-summable since it has dimension $d' < d$.

Let $\beta : R^2 \to R$ be a C^∞ function so that $\beta \equiv 0$ on $\{(x,y) : y < 0 \text{ and } y > 1\}$ and $\beta \equiv 1$ on a small neighborhood V of γ.

Now define $\omega = \beta df$. Note that $\omega \equiv 0$ outside U, and since $\omega = df$ on γ, we thus have $\omega|_{\partial\Omega} = 0$.

It remains to check that ω is d-flat and $\int_{\partial\Omega} \omega \neq 0$.

First, observe that there are constants N, ϵ, depending only on V, such that any 2-simplex τ can be subdivided into at most N sub-simplices $\tau_1, \dots, \tau_j, j \leq N$, such that each τ_i is either contained in V or else has distance at least ϵ from γ.

Furthermore, since ω is C^∞ away from γ, there is a constant K so that $|d\omega(x)| \leq K$ whenever $d(x,\gamma) \geq \epsilon$.

Now

$$|\int_{\partial\tau} \omega| \leq \sum_{i=1}^{j} |\int_{\partial\tau_i} \omega| = \sum_{\tau_i \subset V} |\int_{\partial\tau} \omega| + \sum_{\tau_i \not\subset V} |\int_{\tau_i} d\omega|,$$

where we have made use of the ordinary Gauss-Green theorem in the last sum.

The terms in the first sum on the right are zero since, in V, $\omega = df$ is exact. The terms in the second sum are bounded by K area$(\tau_i) \leq KM_2(\tau) \leq KM_d(\tau)$, so

$$|\int_{\partial\tau} \omega| \leq NKM_d(\tau).$$

This means $||\omega|| < NK$, and hence $|\omega|^d < \infty$.

To show that $\int_{\partial\Omega} \omega \neq 0$, we show that $\int_{\partial W_k} \omega = 1$ for large k, where W_k is the polyhedral approximation to Ω given by the Whitney decomposition.

Divide the set ∂W_k into two parts $\partial W_k = S_k \cup T_k$, where $S_k = \partial W_k \setminus U$ and $T_k = \partial W_k \cap U$.

Now $\int_{S_k} \omega = 0$ since $\omega = 0$ off U. For large k, $T_k \subset V$, so $\int_{T_k} \omega = \int_{T_k} df = 1$, since the endpoints of T_k all lie on $x = 0$ or $x = 1$, where f has value 0 or 1, respectively.

Hence $\int_{\partial W_k} \omega = \int_{S_k} \omega + \int_{T_k} \omega = 1$. //

We can do away with the summability assumption in the Lipschitz case:

Theorem A'. *If ω is H^{n-1}-equal to a Lipschitz form and Ω is any Jordan domain in R^n (even with positive measure boundary), then*

$$\omega|_{\partial\Omega} = 0 \textit{ implies } \int_{\partial W_k} \omega \to 0,$$

where W_k is the polyhedral approximation determined by the Whitney decomposition of Ω.

Proof. The Whitney decomposition of any bounded region is always n-summable. Now repeat the proof of Theorem A with $d = n$. //

Even when ω is H^{n-1}-a.e. differentiable on Ω and $d\omega \in L^1(\Omega)$, we cannot expect to have a Gauss-Green Theorem, even for rectangles, because of the usual Cantor function counterexamples to the Fundamental Theorem of Calculus. In one variable, this motivates the notion of absolute continuity. In higher dimensions, the natural counterpart is the "ACL" condition:

Definition. *A function f is ACL (absolutely continuous on lines) in a domain Ω if , in each closed rectangle $R \subset \Omega$, f is absolutely continuous as a function of one variable when restricted to almost every line parallel to each coordinate axis.*

A form is ACL if each of its component functions is ACL.

Note: An ACL function has finite partial derivatives a.e. [LV], so that $d\omega$ is defined pointwise a.e. if ω is ACL.

Definition. *We denote by $W^{1,1}(\Omega)$ the Sobolev space of $(n-1)$-forms ω satisfying*
(i) ω is ACL in Ω, and
(ii) $d\omega \in L^1(\Omega)$.

This space $W^{1,1}(\Omega)$ is a natural space of forms for which Green's formula holds for rectangular subregions of Ω [LV, lemma III.6.1]. The benefit of sections 2 and 3 is that we now have a meaningful definition for the expression on the LHS of (1), logically independent of the RHS. The theorem is the following:

Theorem B (Gauss-Green). *Let $d \in (n-1, n]$. If $\omega \in F^d \bigcap W^{1,1}(\Omega)$, and Ω is a Jordan domain in R^n such that $\partial\Omega$ is d-summable,*
then $\int_{\partial\Omega} \omega = \int_\Omega d\omega$.

Proof. The main work has already been done:
If $\mathcal{W}$ is the Whitney decomposition of Ω, then $\int_\Omega d\omega = \lim \int_{\partial W_k} \omega$.

Since $\omega \in W^{1,1}(\Omega)$, we have, for any Whitney cube Q, $\int_Q d\omega = \int_{\partial Q} \omega$ by the standard arguments via the Fundamental Theorem of Calculus.

Hence $\int_{W_k} d\omega = \int_{\partial W_k} \omega$. Now by the Lebesgue Dominated Convergence Theorem, the left hand side above tends to $\int_\Omega d\omega$. //

Appendix

Proof of Proposition 2.
We are given $\overline{\Omega} = \sum_1^N \sigma_i$ such that $\Omega = \operatorname{int}(\cup\sigma_i)$, and we wish to show that

$$M_d(\overline{\Omega} - W_k) \to 0 \text{ as } k \to \infty.$$

Let $W(k)$ denote the union of the simplices of W_k.

Fix k large enough that $\mathcal{W}^k \neq \emptyset$.

For $j \geq k$, define the collections $\mathcal{U}(j)$ and $\mathcal{V}(j)$ as follows:

$$\mathcal{U}(j) = \bigcup_{k<i\leq j} \mathcal{W}^i, \text{ and}$$

$$\mathcal{V}(j) = \{Q \cap \Omega : Q \text{ is a } j\text{-cube, and } Q \not\subseteq \cup\mathcal{U}(j)\}.$$

Note that $\mathcal{U}(j) \cup \mathcal{V}(j)$ is a finite decomposition of $\Omega \setminus W(k)$.

Since every element of $\mathcal{V}(j)$ is contained in the $2^{-j+2}\sqrt{n+1}$-neighborhood of $\partial\Omega$, the cardinality of $\mathcal{V}(j)$ is at most $C2^{j(n-1)}$, where C is some constant depending on Ω but not on j or k.

Choose the constant $C' = C'(n)$ so large that for any n-cube Q and n-simplex σ, $Q \cap \sigma$ can be subdivided into at most C' n-simplices. Then for any n-cube Q, $Q \cap \Omega$ can be subdivided into at most NC' n-simplices.

Now we may subdivide each element of $\mathcal{U}(j)$ and $\mathcal{V}(j)$ into at most NC' affine n-simplices. Denote the collection of all such simplices by $\mathcal{T}(j)$; this is a finite affine decomposition of $\Omega \setminus W(k)$, and so its d-sum estimates $M_d(\overline{\Omega} - W_k)$.

The d-sum of $\mathcal{T}(j)$ is at most

$$NC' \left\{ \left(\sum_{k<i\leq j} \sum_{Q\in\mathcal{W}^i} |Q|^d \right) + C2^{j(n-1)}2^{-jd}(n+1)^{d/2} \right\}$$

$$\leq C'' \left\{ \left(\sum_{i>k} \sum_{Q\in\mathcal{W}^i} |Q|^d \right) + 2^{j(n-d-1)} \right\}$$

where C'' depends only on n and Ω.

Letting $j \to \infty$, this shows that

$$M_d(\overline{\Omega} - W_k) \leq C'' \sum_{i>k} \sum_{Q\in\mathcal{W}^i} |Q|^d,$$

which tends to zero as $k \to \infty$ by Lemma 2. //

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