DENJOY MINIMAL SETS ARE FAR FROM AFFINE

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Abstract. Not every Cantor set can arise as the minimal set for a C^1 diffeomorphism of the circle. For example, D. McDuff has shown that the usual "middle thirds" Cantor set cannot arise this way. In this note we exclude a suitable neighborhood of the class of affine Cantor sets.

1. Introduction

In [7], D. McDuff addresses the following question (attributed to M. Herman): for which Cantor subsets C of the circle $\mathbf{S}^1$ does there exist a C^1 diffeomorphism of $\mathbf{S}^1$ having minimal set C? (See below for definitions.)

To broaden the question a little, for $r \geq 0$ we will denote by $\mathcal{C}(r)$ the class of C^r-*minimal* sets; that is,

$$\mathcal{C}(r) = \{C \subset \mathbf{S}^1 : C \text{ is a minimal Cantor set for some } C^r \text{ diffeomorphism of } \mathbf{S}^1\}.$$

Since any two Cantor sets in S^1 are homeomorphic by a homeomorphism of S^1, it is easy to see that $\mathcal{C}(0)$ includes every Cantor set. Moreover, Denjoy's theorem, stated below, implies that $\mathcal{C}(r)$ is empty for $r \geq 2$. Other cases are more subtle. There are partial answers to Herman's question about $\mathcal{C}(1)$. McDuff gives several necessary conditions for membership in $\mathcal{C}(1)$, one of which implies that the usual "middle thirds" Cantor set does *not* belong to $\mathcal{C}(1)$.

The purpose of this note is to establish that all affine and C^2-nearly affine Cantor sets are excluded from $\mathcal{C}(1)$. To make this precise, we need to introduce some terminology.

Let $I_1, \dots, I_k$, $k \geq 2$, be pairwise disjoint compact intervals in $\mathbf{R}$, and let L be a compact interval containing their union $I \equiv I_1 \cup \dots \cup I_k$.

Define $\mathcal{S}^r(I_1, \dots, I_k, L)$ to be the set of C^r functions $S : I \to L$ such that $|S'| > 1$ on I and, for each $j = 1, \dots, k$, $S[I_j] = L$.

Any $S \in \mathcal{S}^r(I_1, \dots, I_k, L)$ has a unique maximal invariant (Cantor) set

$$C_S = \{x \in I : S^k(x) \in I \text{ for all } k \in \mathbf{Z}^+\}.$$

A Cantor set arising this way is called *hyperbolic*. If S can be chosen so that $|S'|$ is locally constant, the Cantor set is called *affine*. If $|S'|$ is globally constant, we say C_S is a *linear* Cantor set. Of course every linear Cantor set is affine, and every affine Cantor set is hyperbolic.

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For a simple example, let $k = 2$, $I_1 = [0, 1/3]$, $I_2 = [2/3, 1]$, and $L = [0, 1]$. Define S by $S(x) = 3x$ for $x \in I_1$ and $S(x) = 3x - 2$ for $x \in I_2$. Then the linear Cantor set C_S is the familiar "middle thirds" Cantor set.

To discuss hyperbolic Cantor sets in the circle S^1, we take $S^1 = \mathbf{R}/\mathbf{Z}$, work in coordinates $[0, 1)$, and always suppose $L \subset [0, 1)$. (The middle thirds Cantor set would have to be slightly scaled down to fit.)

We now state the main results.

Theorem 1. *No affine Cantor set is C^1-minimal.*

Theorem 2. *Let $I_1, \ldots, I_k, L$ be compact intervals in $[0, 1)$ as above.*

Then there exists $\epsilon > 0$ (depending only on $\{|I_j|/|L| : j = 1, \ldots, k\}$) such that if $S \in \mathcal{S}^2(I_1, \ldots, I_k, L)$ and

$$\mathcal{N}(S) \equiv \max_{j=1,\ldots,k} \sup_{x,y \in I_j} \log \frac{S'(x)}{S'(y)} < \epsilon,$$

then C_S is not C^1-minimal.

Remarks. 1. Theorem 1 follows immediately from Theorem 2 since $\mathcal{N}(S) = 0$ when S is affine.

2. The C^2 hypothesis in the Theorem 2 can actually be replaced by C^{1+Lip} with the same proof. Provided one is willing to impose a uniform Hölder bound on S, the smoothness hypothesis can be improved further to $C^{1+Zygmund}$, $C^{1+b.v.}$, or any smoothness class for which Denjoy's Theorem (see below) holds true.

3. The reader may wonder whether the condition of Theorem 2 is the best way to interpret the meaning of "nearly affine" Cantor set. A very natural alternative is as follows. Let $\mathcal{K}$ denote the class of all Cantor sets in $\mathbf{S}^1$. If G is a group of circle homeomorphisms with metric d, then G acts naturally on $\mathcal{K}$ and

$$d_G(K_1, K_2) \equiv \inf\{d(g, id) : g \in G \text{ and } g(K_1) = K_2\}$$

provides a metric on each G-orbit in $\mathcal{K}$, and therefore a topology on $\mathcal{K}$. However, this kind of topology does not seem well-suited to our problem.

For example, if G is the full homeomorphism group (acting transitively on $\mathcal{K}$) with the C^0 metric, then the topology induced by d_G is too coarse (insensitive to geometry) and the analog of Theorem 2 is false.

If instead we take G to be the group of C^1 diffeomorphisms with the C^1 metric, then the topology is much finer than the one indicated in Theorem 2. Moreover the collection of C^1-minimal Cantor sets is trivially a union of full G-orbits, so the metric gives no extra information. Such considerations should make the notion of "nearly affine" in Theorem 2 seem more natural.

Theorems 1 and 2 leave many open questions, the most immediate one being

1. Does $\mathcal{C}(1)$ contain any hyperbolic Cantor sets at all?

More generally,

2. Is there a purely geometric characterization of those Cantor sets belonging to $\mathcal{C}(1)$?

3. What can be said for $\mathcal{C}(\alpha)$, $0 < \alpha < 2$, $\alpha \neq 1$?

Some things are already known in relation to question 3. It is shown in [10] that any 2-branched linear Cantor set in S^1 is the minimal set for some bi-Lipschitz homeomorphism. This means the "C^1" of Theorem 1 cannot be weakened even to "Lipschitz". Also, in [9] it is shown that if $C \in \mathcal{C}(1+\epsilon), 0 < \epsilon < 1$, then the upper box dimension (upper Minkowski content) of C must be at least ϵ.

A number of mathematicians, including J. Harrison, Y. Katznelson, and B. Kra, have considered the Hausdorff dimension of $C \in \mathcal{C}(r)$ in relation to r. Here the Diophantine type of the rotation number becomes highly important. This is work in progress.

In the next two sections we will introduce some terminology and standard results. The proof of Theorem 2 is given in Section 4. In Section 5 we state the main theorem of McDuff to show that it does not already subsume Theorem 1.

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2. Minimal sets in $\mathbf{S}^1$

If f is a homeomorphism, the set Γ is a *minimal set* for f if Γ is compact, nonempty, invariant, and minimal (relative to inclusion) with respect to these three properties. Equivalently, $\Gamma \neq \emptyset$ is minimal if $f(\Gamma) = \Gamma$ and every f-orbit in Γ is dense in Γ.

The simplest examples of minimal sets are fixed points or periodic orbits. Zorn's Lemma implies that every compact orbit closure contains some minimal set, so every homeomorphism of a compact manifold has at least one minimal set.

For homeomorphisms of $\mathbf{S}^1$, Poincaré already understood all the possibilities. Either f has a periodic orbit, in which case all its minimal sets are finite, or else f has no periodic orbits, in which case f has a unique minimal set which is either $\mathbf{S}^1$ itself (the transitive case) or a Cantor set C (the intransitive case). In the transitive case, f is topologically conjugate to an irrational rotation. In the intransitive case, C is the ω-limit set of every point, and f is semiconjugate to an irrational rotation R, i.e. $hf = Rh$ where h is a continuous monotone function with $h(C) = \mathbf{S}^1$ (a "Cantor function").

The intransitive case can be realized as a C^1 diffeomorphism (Bohl [2]), often called a *Denjoy counterexample* because of Denjoy's theorem, which states that such a diffeomorphism cannot be too smooth:

Denjoy's Theorem [3]. *If f is a C^1 diffeomorphism of $\mathbf{S}^1$ without periodic points, and if the derivative Df has bounded variation, then f is topologically conjugate to an irrational rotation.*

(There are more recent results along these lines. Herman [5] has produced counterexamples of class $C^{1+\alpha}$ for all $\alpha < 1$. Sullivan and Hu [6] have shown, for example, that the bounded variation condition in Denjoy's theorem can be replaced with a Zygmund condition. Hall [4] has shown that there are C^∞ homeomorphisms conjugate to Denjoy counterexamples, and Yoccoz [12] proved that every real analytic homeomorphism without periodic points is conjugate to an irration rotation. See also de Melo-van Strien [8].)

We can formulate some of the above conveniently as follows:

The Poincaré-Denjoy dichotomy. *If f is a C^2 diffeomorphism of $\mathbf{S}^1$ and Γ is a minimal set for f, then either Γ is finite or $\Gamma = \mathbf{S}^1$.*

3. Hyperbolic Cantor sets

If a Cantor set in $[0,1)$ can be realized as C_S for some $S \in \mathcal{S}^r(I_1, \dots, I_k, L)$, it is called a C^r *hyperbolic Cantor set.* Every Cantor set is C^0 hyperbolic (for $S \in C^0$ we replace the condition $|S'| > 1$ with the requirement that S be locally strictly monotone). For $r > 1$, every C^r hyperbolic Cantor set has Lebesgue measure zero. (In this case there is already a rich literature on the structure of these sets; see for example Bedford-Fisher [BF], or Sullivan [S].)

For simplicity, we will restrict attention in the remainder of this note to studying $\mathcal{S}^r(I_1, \dots, I_k, L)$ in the case when $k = 2$, and L is the convex hull of I_1 and I_2. Furthermore we consider only those elements S that are everywhere increasing. (The proof is essentially the same, but more complicated, if these assumptions are relaxed.) Moreover, we may assume, by global change of coordinates if necessary, that one of the complementary intervals of C_S is $(1/2, 1)$ (so that $L = [0, 1/2]$). Defining $I_1 = [0, a]$ and $I_2 = [b, 1/2]$, where $0 < a < b < 1/2$, the graph of S is then as shown in the following figure.

[Figure]

The following notation will be convenient. Let $\phi_0 : [0, 1/2] \to [0, a]$ and $\phi_1 : [0, 1/2] \to [b, 1/2]$ be the two branches of the inverse of S.

For any choice $i_1, \dots, i_k \in \{0, 1\}$, write $\phi(i_1, \dots, i_k)$ for the composition $\phi_{i_1} \circ \dots \circ \phi_{i_k}$. We will call the interval $\phi(i_1, \dots, i_k)([0, 1/2])$ a *k-block* of C. (Then if C_k denotes the union of the 2^k disjoint k-blocks, $C_S = \bigcap C_k$.)

Let $\mathcal{I}$ denote the collection of "gap" intervals of C, that is, the collection of connected components of $[0, 1/2] \setminus C_S$. We set $I_0 = (a, b) \in \mathcal{I}$.

If $I \in \mathcal{I}$, then for some $j \in \mathbf{Z}^+$ and $i_1, \dots, i_j \in \{0, 1\}$, we have

$$I = \phi(i_1, \dots, i_j)(I_0).$$

We then denote by $bl(I)$ the block $\phi(i_1, \dots, i_j)([0, 1/2])$.

Given such $I = \phi(i_1, \dots, i_j)(I_0)$ and any $j_1, \dots, j_n \in \{0, 1\}$, let $I(j_1, \dots, j_n)$ denote the interval

$$\phi(i_1, \dots, i_k, j_1, \dots, j_n)(I_0) \subset bl(I)$$

and let $bl(I)(j_1, \dots, j_n)$ denote the interval

$$\phi(i_1, \dots, i_k, j_1, \dots, j_n)([0, 1/2]) \subset bl(I).$$

For example, $bl(I)(0)$ is the left subblock of $bl(I)$, and $I(0)$ is its "middle" interval.

We conclude this section with the following standard fact.

Bounded Distortion Lemma. *Suppose $S \in \mathcal{S}^{1+\beta}(I_1, I_2, L)$ for some $\beta \in (0,1]$. For every $\epsilon > 0$ there exists $N \in \mathbf{Z}^+$ such that for all $m \geq N$,*
if J is any m-block of C_S and $x, y \in J$, then

$$|\log \frac{(S^{m-N+1})'(x)}{(S^{m-N+1})'(y)}| < \epsilon.$$

Proof. Since $\log S'$ is β-Hölder, we can find $M > 0$ so that $|\log S'(x) - \log S'(y)| \leq M|x-y|^\beta$. Also, there is $\alpha \in (0,1)$ such that $|S'| \geq 1/\alpha$.

For any $k < m$, the Mean Value Theorem implies that

$$|S^{k+1}(x) - S^{k+1}(y)| \geq (1/\alpha)|S^k(x) - S^k(y)|,$$

and so $|S^k(x) - S^k(y)| \leq \alpha |S^{k+1}(x) - S^{k+1}(y)|$.

Inductively, for $k < m$,

$$|S^k(x) - S^k(y)| \leq \alpha^{m-k}|S^m(x) - S^m(y)| \leq \alpha^{m-k}.$$

Making use of the chain rule, the Hölder condition on $\log S'$, and this estimate, we have

$$\begin{aligned}
|\log \frac{(S^{m-N+1})'(x)}{(S^{m-N+1})'(y)}| &= |\log (S^{m-N+1})'(x) - \log (S^{m-N+1})'(y)| \\
&= |\sum_{i=0}^{m-N} \log S'(S^i(x)) - \log S'(S^i(y))| \\
&\leq \sum_{i=0}^{m-N} M|S^i(x) - S^i(y)|^\beta \\
&\leq \sum_{i=0}^{m-N} M(\alpha^{m-i})^\beta = M \sum_{j=N}^{m} \alpha^{\beta j} \\
&\leq M \sum_{j=N}^{\infty} \alpha^{\beta j} = M \frac{\alpha^{\beta N}}{1-\alpha^\beta},
\end{aligned}$$

and this can be made as small as desired by suitable choice of N (and we note that N depends only on M, β, and α).

4. Proof of Theorem 2

For a diffeomorphism $f : J \to K$ between intervals J and K, we define the *nonlinearity* of f to be

$$\mathcal{N}(f) = \sup_{x,y \in J} \log |\frac{f'(x)}{f'(y)}|.$$

Lemma 1. *If E, F, G are intervals and $f : E \to F$ and $g : F \to G$ are diffeomorphisms, then*

$$\mathcal{N}(f) = \mathcal{N}(f^{-1}) \text{ and } \mathcal{N}(g \circ f) \leq \mathcal{N}(f) + \mathcal{N}(g).$$

Proof. The proof is a simple application of the inverse function theorem and the chain rule.

We need some further notation. For any $I, J \in \mathcal{I}$, there exist unique $k \in \mathbf{Z}^+$ and $i_1, \ldots, i_j \in \{0, 1\}$ such that $S^k(I) = I_0$ and $\phi(i_1, \ldots, i_j)(I_0) = J$. Define

$$\Phi_{I,J} = \phi(i_1, \ldots, i_j) \circ S^k.$$

Then $\Phi_{I,J}(I) = J$, $\Phi_{I,J}(bl(I)) = bl(J)$, and $\Phi_{I,J}$ is as smooth as S.

For any $I \in \mathcal{I} \setminus \{f^{-1}((1/2, 1))\}$, we can now define

$$\Phi_I = \Phi_{I,f(I)}.$$

Lemma 2. *Given I_1, I_2, and L as before, there exists $\epsilon > 0$ such that if $S \in \mathcal{S}^2(I_1, I_2, L)$ and $\mathcal{N}(S) < \epsilon$, then for all $I \in \mathcal{I}$ sufficiently small,*

$$f|_{C \cap bl(I)} = \Phi_I|_{C \cap bl(I)}.$$

Proof. Let A be the affine, locally increasing member of $\mathcal{S}^2(I_1, I_2, L)$. Let

$$\alpha_1 = \max\{|I_1|/|L|, |I_2|/|L|\} \in (0, 1).$$

In particular, this means $A' \geq 1/\alpha_1$.

Choose α and α_2 so that $\alpha_1 < \alpha_2 < \alpha < 1$. We may now choose $\epsilon_0 > 0$ so small that the following three conditions are satisfied:

(i) $\alpha \exp(\epsilon_0) < 1$,
(ii) $\alpha_2 \exp(\epsilon_0) < \alpha$, and
(iii) $\alpha_1 \exp(\epsilon_0) < \alpha_2$.

It is straightforward to verify, by virtue of (iii), that for each $S \in \mathcal{S}^2(I_1, I_2, L)$, $\mathcal{N}(S) < \epsilon_0$ implies $S' \geq 1/\alpha_2$.

By the Bounded Distortion Lemma, we may find $N > 1$ so that $\mathcal{N}(S^{m-N}|_{J(m)}) < \epsilon_0/8$ for all $m > N$ and all m-blocks $J(m)$.

Now let $\epsilon = \epsilon_0/4N$, and fix $S \in \mathcal{S}^2(I_1, I_2, L)$ so that $\mathcal{N}(S) < \epsilon$. For any m-block I and n-block J, where $m, n > N$, we have

$$\mathcal{N}(\Phi_{I,J}) = \mathcal{N}((\phi_{i_n} \circ \cdots \circ \phi_{i_{N+1}}) \circ (\phi_{i_N} \circ \cdots \circ \phi_{i_1}) \circ S^N \circ S^{m-N}|_I)$$

$$\leq \epsilon_0/8 + N(\epsilon_0/4N) + N(\epsilon_0/4N) + \epsilon_0/8 = 3\epsilon_0/4,$$

where we have used Lemma 1 for the last inequality.

Since f is C^1, we may choose m so large that if J is any m-block of C_S, then $\mathcal{N}(f|_J) < \epsilon$. Again by Lemma 1, choosing the block I small enough that $f(I)$ is contained in some m-block, we have

$$\mathcal{N}(f^{-1} \circ \Phi_I) \leq \epsilon + 3\epsilon_0/4 < \epsilon_0.$$

Define $h : I \to [0,1)$ by $h(x) = f^{-1} \circ \Phi_I(x)$. From the definition of Φ_I, it follows that $h(I) = I$, so there is a point $y \in I$ such that $h'(y) = 1$. Since $\mathcal{N}(h) < \epsilon_0$,

$$\exp(-\epsilon_0) < |h'(x)| < \exp(\epsilon_0)$$

for all $x \in I$. By our choice of ϵ_0, we know $\alpha < \exp(-\epsilon_0)$, so

$$(*) \qquad \alpha < |h'(x)| < 1/\alpha$$

for all $x \in I$.

Now choose and fix $I \in \mathcal{I}$ sufficiently small that (*) holds; in particular, so that $bl(I)$ is an m-block for $m > N$ as above.

Claim. If $J \in \mathcal{I}$ and $J \subset bl(I)$, then $h(J) = J$.

It follows from this Claim that $h(x) = x$ for all $x \in bl(I) \cap C_S$, and this is the conclusion of Lemma 2.

The Claim is proved by induction on the "level" of the gaps in $bl(I)$, as follows.

First, we have noted that $h(I) = I$. Next, we may suppose by induction that for $k > 0$ and all choices $i_1, \ldots, i_k \in \{0,1\}$,

$$h(I(i_1, \ldots, i_k)) = I(i_1, \ldots, i_k).$$

Let $J = I(i_1, \ldots, i_k)$ for some particular choice of $i_1, \ldots, i_k$.

We show that $h(J(0)) = J(0)$. (The case $J(1)$ is similar.)

Now, for some n, $S^n(J) = (a,b)$, so

$$\begin{aligned} \frac{|J(0)|}{|J|} &\le (\max_{x,y \in J} \frac{(S^n)'(x)}{(S^n)'(y)}) \frac{|\phi_0[(a,b)]|}{|(a,b)|} \\ &\le (\exp(\mathcal{N}(S^n|_J)))\alpha_2 \le (\exp(\epsilon_0))\alpha_2 < \alpha. \end{aligned}$$

Similarly,

$$\frac{|J(0, i_1, \ldots, i_l)|}{|J(0, i_1, \ldots, i_{l-1})|} < \alpha$$

for all l, $i_1, \ldots, i_l \in \{0,1\}$.

Case 1. $h(J(0)) \not\subseteq bl(J(0))$.

Since $h(J) = J$, the intermediate value theorem implies there is some interval $K = J(0, 1, j_1, \ldots, j_s)$ such that $h(K) = J(0)$. By the estimate above, K is smaller than $J(0)$ by at least a factor α, and this contradicts (*).

Case 2. $h(J(0)) \subset bl(J(0))$. If $h(J(0)) \neq J(0)$, then $|h(J(0))| \le \alpha |J(0)|$, since all other $\mathcal{I}$ intervals in $bl(J(0))$ are smaller than $J(0)$ by at least this factor. This again contradicts (*), leaving $h(J(0)) = J(0)$ as the only remaining possibility.

Proof of Theorem 2.

Suppose for contradiction that f is a C^1 diffeomorphism of $\mathbf{S}^1$ and that its (unique) minimal set C is equal to C_S, where S is chosen with $\mathcal{N}(S) < \epsilon$ as in the proof of Lemma 2.

We can cover C, by compactess, with finitely many small blocks $bl(I)$ so that, on each one,

$$f|_{C \cap bl(I)} = \Phi_I|_{C \cap bl(I)}.$$

That is, f agrees on C with a C^2 function defined on a finite union of disjoint compact intervals. Since f is monotone, this C^2 function can be extended to a C^2 diffeomorphism g of $\mathbf{S}^1$.

Since $g|_C = f|_C$, every g-orbit in C is dense in C, so the set C must be a minimal set for g. This contradicts the Poincare-Denjoy dichotomy.

5. McDuff's Theorem

In this section we state the relevant theorem of [7], and show it does not imply Theorem 1.

First we need some notation. Let $\lambda_1 \geq \lambda_2 \geq .... > 0$ be the lengths of the complementary intervals of C in $\mathbf{S}^1$; the set of such lengths is the *length spectrum* $\sigma(C)$ of C.

The theorem of McDuff gives a necessary condition for membership in $\mathcal{C}(1)$ purely in terms of the length spectrum of the Cantor set.

Choose any sequence of disjoint subintervals $\{J_i = [\alpha_i, \beta_i] : i = 1, 2, 3, \ldots\}$ subject to the requirements that $\alpha_{i+1} \leq \beta_{i+1} < \alpha_i$ for all i, and

$$\sigma(C) \subset \bigcup_{k=1}^{\infty} J_k.$$

Spectral Theorem [McDuff]. *Suppose λ_i, α_j and β_j are defined as above and for each $N > 0$ there is $\eta(N) > 0$ such that for all $n \in [-N, N]$ and all $j > N$,*

$$\frac{\alpha_{j+n-1}}{\beta_{j+n}} \geq (1+\eta)\frac{\beta_j}{\alpha_j}.$$

Then C is not C^1-minimal.

Corollary. *If C is C^1-minimal, then the ratios λ_i/λ_{i+1} are bounded and have 1 as a nontrivial limit point.*

The corollary follows from the theorem by setting $\alpha_i = \beta_i = \lambda_i$ for all i and applying the case $n = 1$. In turn, the corollary implies that the usual middle thirds Cantor set is not C^1-minimal, because for that set the ratios λ_i/λ_{i+1} take only the values 3 and 1. The same goes for any affine Cantor set as described in Figure 1 so long as the slopes of the two branches are equal, i.e. $a = 1/2 - b$, or equivalently, $l = r$.

However, the hypotheses on the length spectrum in the above theorem are generically not satisfied by an affine Cantor set (and so Theorem 1 does not follow from the Spectral Theorem). This is not obvious from the statement, so we formalize the matter in the following proposition.

First note that if $l \neq r$, then $\{l^i r^j (b-a) : i, j \in \mathbf{Z}^+\} \subset \sigma(C)$. The above claim therefore follows from the

Proposition. *Suppose $x, y \in (0,1)$ and, for all $m, n \in \mathbf{Z}$, $x^m \neq y^n$. Fix $a > 0$ and let $S = \{x^i y^j a : i, j \in \mathbf{Z}^+\}$.*

Choose any positive sequences $\{\alpha_i\}$, $\{\beta_i\}$ such that

(i) $\alpha_i \to 0$ and $\beta_i \to 0$,

(ii) $\alpha_{i+1} \leq \beta_{i+1} \leq \alpha_i$ for all i, and

(iii) $S \subset \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i]$.

Then for all $\eta > 0$ and $N \in \mathbf{Z}^+$ there exists $j > N$ such that

$$\frac{\alpha_j}{\beta_{j+1}} < (1+\eta)\frac{\beta_j}{\alpha_j}.$$

Proof. Since the conclusion is unchanged by a uniform scaling, we assume for convenience that $a = 1$ and

$$S = \{x^i y^j : i, j \in \mathbf{Z}^+\}.$$

Let $\eta > 0$ and $N > 0$ be given. Choose $m, n \in \mathbf{Z}$ so that $r \equiv x^m y^n \in (1/(1+\eta), 1)$. Then choose $k \in \mathbf{Z}^+$ so that $r^k < x$, and define $p = |m|k + 1, q = |n|k + 1$, and $s = x^p y^q \in S$.

By hypothesis (iii), for some j, $s \in [\alpha_j, \beta_j] \equiv J$. There are now two cases.

Case 1. $r^k s \in J$.

Then

$$\frac{\beta_j}{\alpha_j} \geq \frac{s}{r^k s} = \frac{1}{r^k} > \frac{1}{x}.$$

On the other hand, for all i, $\alpha_i/\beta_{i+1} < 1/x$. (This is because of (iii) and the fact that $z \in S$ implies $xz \in S$.)

Hence

$$\frac{\alpha_j}{\beta_{j+1}} < \frac{1}{x} < \frac{\beta_j}{\alpha_j} < (1+\eta)\frac{\beta_j}{\alpha_j}.$$

Case 2. $r^k s \notin J$.

Then there is an integer l, $0 < l \leq k$, such that $r^{l-1}s \in J$ but $r^l s \notin J$. Then

$$\frac{\alpha_j}{\beta_{j+1}} \leq \frac{r^{l-1}s}{r^l s} = \frac{1}{r} < (1+\eta) \leq (1+\eta)\frac{\beta_j}{\alpha_j}.$$

□

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