DENJOY'S THEOREM WITH EXPONENTS

Alec Norton

Abstract. If X is the (unique) minimal set for a $C^{1+\alpha}$ diffeomorphism of the circle without periodic orbits, $0 < \alpha < 1$, then the upper box dimension of X is at least α. The method of proof is to introduce the exponent α into the proof of Denjoy's theorem.

1. Introduction

If f is a homeomorphism, the set Γ is a *minimal set* for f if Γ is compact, non-empty, invariant, and minimal (relative to inclusion) with respect to these three properties. Equivalently, $\Gamma \neq \emptyset$ is minimal if $f(\Gamma) = \Gamma$ and every f-orbit in Γ is dense in Γ.

The simplest examples of minimal sets are fixed points or periodic orbits. Zorn's Lemma implies that every homeomorphism of a compact manifold has at least one minimal set.

In this paper we consider homeomorphisms of the circle, where Poincaré [13] already understood all the possibilities. Either f has a periodic orbit, in which case all its minimal sets are finite, or else f has no periodic orbits, in which case f has a unique minimal set which is either $\mathbf{S}^1$ itself (the transitive case) or a Cantor set C (the intransitive case). In the transitive case, f is topologically conjugate to an irrational rotation. In the intransitive case, C is the set of accumulation points of the forward f-orbit of every point. Moreover, for each interval I disjoint from C, $f^n(I) \cap f^m(I) = \emptyset$ for $n \neq m$.

The intransitive case can be realized as a C^1 diffeomorphism (Bohl [3]). This is often called a *Denjoy counterexample* because of Denjoy's theorem, which states that such a diffeomorphism cannot be too smooth:

Denjoy's Theorem [4]. *If f is a C^1 diffeomorphism of $\mathbf{S}^1$ without periodic points, and if the derivative Df has bounded variation (e.g. if f is C^2), then f is topologically conjugate to an irrational rotation.*

Herman [8] produced Denjoy counterexamples of class $C^{1+\alpha}$ for all $\alpha < 1$. He also [private communication] raised the following

Question. for which Cantor subsets C of the circle $\mathbf{S}^1$ does there exist a C^1 diffeomorphism of $\mathbf{S}^1$ having minimal set C?

1991 *Mathematics Subject Classification.* Primary 58F03, Secondary 28A80.

Key words and phrases. box dimension, minimal sets, Cantor sets, circle diffeomorphisms.

For $r \geq 0$, we will denote by $\mathcal{C}(r)$ the class of C^r*-minimal* sets; that is,

$$\mathcal{C}(r) = \{C \subset \mathbf{S}^1 : C \text{ is a minimal Cantor set for some } C^r \text{ diffeomorphism of } \mathbf{S}^1\}.$$

Since any two Cantor sets are homeomorphic, it is easy to see that $\mathcal{C}(0)$ includes every Cantor set. Moreover, Denjoy's theorem implies that $\mathcal{C}(r)$ is empty for $r \geq 2$. Other cases are more subtle.

There are partial answers to Herman's question about $\mathcal{C}(1)$. For example, the usual middle thirds Cantor set does NOT belong to $\mathcal{C}(1)$. See McDuff [9], Norton [11] for more results excluding certain Cantor sets. The purpose of this note is to present the proof of the following theorem. Let $BD(X)$ denote the upper box dimension of the set X (defined below).

Theorem 1. *For any $\alpha \in [0, 1)$, if $C \in \mathcal{C}(1 + \alpha)$ then $BD(C) \geq \alpha$.*

This statement is sharp, as shown in section 4 below. The following corollary is immediate.

Corollary 1 (Denjoy's Theorem "with exponents"). *If f is a diffeomorphism of the circle with minimal set C and $BD(C) = \alpha \in (0, 1)$, then $f \notin C^{1+\beta}$ for any $\beta > \alpha$.*

Theorem 1 has a simple corollary for planar diffeomorphisms. Define the ω-limit set $\omega_f(x)$ of a point x for a homeomorphism f to be the set of limit points of the sequence $\{f^n(x) : n \in Z^+\}$.

Corollary 2. *Let K be a $C^{1+\alpha}$ circle in the plane, and let f be a $C^{1+\alpha}$ diffeomorphism of a neighborhood of K into itself such that $f(K) = K$.*

If $f|_K$ has no periodic orbits, then, for all $x \in K$,

$$BD(\omega_f(x)) \geq \alpha.$$

Proof of Corollary 2. There is a $C^{1+\alpha}$ diffeomorphism h defined on a neighborhood of K that takes K to the unit circle S. The restriction of $g = hfh^{-1}$ to S is an ordinary $C^{1+\alpha}$ circle diffeomorphism. If $\omega_g(h(x)) = S$, then $BD(\omega_g(h(x))) = 1$. Otherwise Theorem 1 applies. Either way $\omega_f(x) \geq \alpha$ since box dimension is a diffeomorphism invariant. QED

Remarks. 1. Ideas related to those of Corollary 2 arose in a paper of J. Harrison [7]. There, she constructed a C^2 diffeomorphism f of an annulus with an invariant fractal circle (a quasicircle) on which f is conjugate to a Denjoy counterexample. The diffeomorphism can be made $C^{2+\alpha}$ but this forces the Cantor minimal set to have Hausdorff (and box) dimension $1 + \alpha$. (An important feature of this example is that Df is the identity at each point of the minimal set.)

However, a version of Corollary 2 for smoothness $C^{2+\alpha}$ and dimension $1 + \alpha$ fails to be true without further hypotheses, even for quasicircles in the plane. This is because G.R. Hall [6] has constructed a C^∞ diffeomorphism g of an annulus with a Lipschitz invariant circle on which g is topologically conjugate to a Denjoy counterexample. In Hall's example, the rotation number is required to be well-approximable by rationals (Liouville), while in the Harrison example the rotation number is required to be badly approximable by rationals (e.g. golden mean). It is unknown whether either kind of example exists for all irrational rotation numbers.

2. P. McSwiggen [10] has produced higher dimensional "Denjoy counterexamples" of class C^n on the n-torus, $n \geq 2$. It is tempting to conjecture that a statement similar to Theorem 1 holds there: in dimension n, a $C^{n+\alpha}$ diffeomorphism in the topological conjugacy class of McSwiggen's example must have a minimal set with dimension at least $n - 1 + \alpha$.

For $n = 2$, some geometric restrictions on the possible minimal sets are derived in [12].

The proof of Theorem 1 proceeds in two steps. In section 2, we show that the box dimension of a compact subset of $\mathbf{R}$ is at least α if the "degree α gap sum" is infinite (Proposition 1). Then in section 3, we show that any $C \in \mathcal{C}(1+\alpha)$ must have infinite degree α gap sum (Theorem 2).

In section 4, we show that Theorem 1 is sharp: for every $\alpha \in (0, 1)$ there is a diffeomorphism $f \in C^{1+\alpha}$ such that its minimal set has box dimension α.

Acknowledgement: The author thanks the referee for helpful suggestions.

2. Box dimension

The upper box dimension $BD(X)$ of a bounded nonempty set X in $\mathbf{R}^n$ is defined as follows. For each $\epsilon > 0$ let $N(\epsilon)$ denote the minimal number of ϵ-balls needed to cover X. Then

$$BD(X) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}.$$

That is, $BD(X)$ is roughly the maximum exponential growth rate of $N(\epsilon)$ with respect to $1/\epsilon$. The lower box dimension is defined the same way with the lim inf instead of the lim sup. One convention is to use the term "box dimension" for the common value of the upper and lower box dimensions if they are equal. (Other names for the same value are "capacity dimension", "Minkowski content", "entropy dimension", "Kolmogorov dimension", and "box-counting dimension" .) In this paper we will simply say "box dimension" as short for "upper box dimension", and so we will enjoy the virtue that the box dimension always exists for any bounded nonempty set.

This is a well-known measure of dimension, and has many nice properties. See Falconer [5] for further details and references. For example, the box dimension is monotone with respect to set inclusion, is a Lipschitz invariant, and is equal to the topological dimension for submanifolds. It also agrees with the Hausdorff dimension for self-similar sets. In general, the box dimension of any set is always greater than or equal to the Hausdorff dimension, and inequality is possible even for closed sets.

If X is a compact subset of $\mathbf{R}$ or $\mathbf{R}/\mathbf{Z}$, let $\mathcal{I}(X)$ denote the collection of all bounded connected components of the complement of X, called the *gaps* of X. Each gap in $\mathcal{I}(X)$ is thus an open interval with endpoints in X.

For $0 < \alpha \leq 1$ we define

$$G_\alpha(X) = \sum |I|^\alpha,$$

where the sum is taken over all intervals $I \in \mathcal{I}(X)$. This is called the "degree α gap sum of X". This concept was introduced by Besicovitch and Taylor [2], who also knew the following proposition, essential for our purposes:

Proposition 1. *If $X \subset \mathbf{R}$ is compact and $BD(X) < \alpha$, then $G_\alpha(X) < \infty$.*

It so happens the converse is also nearly true [1]: if X is compact, has measure zero, and $G_\alpha(X) < \infty$, then $BD(X) \leq \alpha$. This means that if X is compact and has Lebesgue measure zero, then

$$BD(X) = \inf\{s : G_s(X) < \infty\}.$$

Proof of Proposition 1. The proof may be found in Bates-Norton [1]; we give the details here for the reader's convenience.

The first remark is that we may calculate $BD(X)$ by restricting attention to the binary decomposition of $\mathbf{R}$. For $k \in \mathbf{Z}$, a *binary k-interval* is an interval of the form $[l/2^k, (l+1)/2^k)$ for some $l \in \mathbf{Z}$. For a bounded $X \subset \mathbf{R}$, let $\nu(k)$ be the number of binary k-intervals that meet X (necessarily finite). The reader may verify that for $2^{-(k+1)} < \epsilon \leq 2^{-k}$,

$$N(\epsilon)/2 \leq \nu(k) \leq 2N(\epsilon),$$

and this means

$$BD(X) = \limsup_{k\to\infty} \frac{\log \nu(k)}{k \log 2}.$$

If $BD(X) < \alpha$, then for sufficiently large k and some $\delta > 0$,

$$\log \nu(k) < (\alpha - \delta) k \log 2,$$

or $\nu(k) < 2^{k(\alpha-\delta)}$. This implies

$$\sum_{k=1}^{\infty} \nu(k) 2^{-k\alpha} < \infty. \tag{1}$$

Let U be the union of the gaps of X, and let $\mathcal{U}$ denote the binary Whitney decomposition of U. That is, $\mathcal{U}$ is a collection of intervals with the following properties:

i. every element of $\mathcal{U}$ is a binary k-interval for some k,

ii. the union of all elements of $\mathcal{U}$ is U, and distinct elements of $\mathcal{U}$ are disjoint, and

iii. for all $I \in \mathcal{U}$,

$$|I| \leq dist(I, \partial U) \leq 4|I|.$$

(See Stein [14] for details of the construction.)

Now it is easy to check that for any gap J, $\sum\{|I|^\alpha : I \in \mathcal{U} \text{ and } I \subset J\}$ is comparable to $|J|^\alpha$, and so $G_\alpha(X) < \infty$ if and only if

$$\sum\{|I|^\alpha : I \in \mathcal{U}\} < \infty. \tag{2}$$

To establish (2), we argue as follows. Let $\mathcal{U}_k$ be the collection of binary k-intervals of $\mathcal{U}$, and $\mathcal{X}_k$ be the collection of binary k-intervals meeting X. (Recall $\mathcal{X}_k$ has cardinality $\nu(k)$.)

For any interval I, let I' denote the interval with length $10|I|$ and the same midpoint as I. Let $\mathcal{X}'_k = \{I' : I \in \mathcal{X}_k\}$.

Then, by property (iii) above, every $I \in \mathcal{U}_k$ is contained in some $J \in \mathcal{X}'_k$, and moreover any $J \in \mathcal{X}'_k$ contains at most 10 elements of $\mathcal{U}_k$. This means

$$\text{cardinality}(\mathcal{U}_k) \leq 10\nu(k).$$

Therefore

$$\sum\{|I|^\alpha : I \in \mathcal{U}\} = \sum_k \sum\{|I|^\alpha : I \in \mathcal{U}_k\}$$

$$= \sum_k (\text{cardinality}(\mathcal{U}_k)) 2^{-k\alpha} \leq \sum_k 10\nu(k) 2^{-k\alpha} < \infty$$

using (1).

3. Proof of Theorem 1

Let $\mathcal{B}^{1+\alpha}$ denote the collection of all $C^{1+\alpha}$ diffeomorphisms f of the circle $\mathbf{R}/\mathbf{Z}$ with irrational rotation number and a Cantor minimal set Γ_f. By Herman [8], $\mathcal{B}^{1+\alpha}$ is nonempty for all $\alpha \in [0,1)$.

Theorem 1 follows immediately from Proposition 1 and

Theorem 2. *If $f \in \mathcal{B}^{1+\alpha}$ then $G_\alpha(\Gamma_f) = \infty$.*

The idea of the proof is simply to adapt the standard Denjoy argument to the exponent α. Here we adapt the Schwarz version of the proof of Denjoy's theorem, as described in Sullivan [15].

For the sake of simplifying the notation our computations will really take place in the cover $\mathbf{R}$ of $\mathbf{R}/\mathbf{Z}$. The choice of lift will not matter, and we use the same letter f to denote it. Write $\Gamma = \Gamma_f$.

We assume for contradiction that $G_\alpha(\Gamma) = G < \infty$. Since Df is α-Hölder and bounded below, $\log Df$ is also α-Hölder. Choose a constant $M > 0$ so that

$$|\log Df(x) - \log Df(y)| \le M|x-y|^\alpha.$$

We denote the closure of a set X by $cl(X)$. Recall $\mathcal{I}(\Gamma)$ is the collection of bounded connected components of the complement of Γ. We need three lemmas.

Lemma 1. *Let I be any interval of $\mathcal{I}(\Gamma)$. For all $x, y \in cl(I)$ and for all $n > 0$,*

$$e^{-MG} \le \frac{Df^n(x)}{Df^n(y)} \le e^{MG}.$$

Proof. For any such x, y,

$$|\log \frac{Df^n(x)}{Df^n(y)}| = |\sum_{i=0}^{n-1} \log Df(f^i(x)) - \log Df(f^i(y))|$$

$$\le \sum_{i=0}^{n-1} M|f^i(x) - f^i(y)|^\alpha \le MG.$$

Lemma 2. *For I as above, and for any $x \in cl(I)$,*

$$\sum_{n=0}^{\infty} (Df^n(x))^\alpha \le \frac{Ge^{MG\alpha}}{|I|^\alpha}.$$

Proof. By the Mean Value Theorem, for each n there is x_n in I so that $|f^n(I)| = |I|Df^n(x_n)$.

So if $x \in cl(I)$, then $Df^n(x) \le e^{MG} Df^n(x_n)$ by Lemma 1, and so

$$\sum Df^n(x)^\alpha \le \sum e^{MG\alpha} Df^n(x_n)^\alpha = \sum e^{MG\alpha}|f^n(I)|^\alpha/|I|^\alpha \le Ge^{MG\alpha}/|I|^\alpha.$$

Lemma 3. *For any $I \in \mathcal{I}(\Gamma)$, any $x \in cl(I)$, and any $C > 1$, there exists a $\delta > 0$ such that for all y and all $n > 0$,*

$$|x - y| < \delta \text{ implies } Df^n(y) \leq CDf^n(x).$$

Proof. Suppose I, x, and C are chosen and fixed. We establish the conclusion by induction on n. The statement for $n = 1$ is immediate by continuity of Df. Suppose by induction that $Df^i(y) \leq CDf^i(x)$ for all y within δ of x, and for $i = 0, 1, 2, \ldots, n-1$.

By shrinking δ if necessary, we may assume without loss of generality that

$$\exp[MC^\alpha \delta^\alpha G e^{MG\alpha}/|I|^\alpha] \leq C. \tag{3}$$

(Note that this condition is independent of n.)

Now we estimate as before:

$$|\log \frac{Df^n(x)}{Df^n(y)}| = |\sum_{i=0}^{n-1} \log Df(f^i(x)) - \log Df(f^i(y))|$$

$$\leq \sum_{i=0}^{n-1} M|f^i(x) - f^i(y)|^\alpha$$

$$= \sum_{i=0}^{n-1} MDf^i(x_i)^\alpha |x - y|^\alpha$$

by the Mean Value Theorem, for some choices x_i between x and y,

$$\leq \sum_{i=0}^{n-1} MC^\alpha Df^i(x)^\alpha \delta^\alpha \quad \text{by induction}$$

$$\leq MC^\alpha \delta^\alpha G e^{MG\alpha}/|I|^\alpha \quad \text{by Lemma 2.}$$

Now taking exponentials and applying (3) yields $Df^n(y) \leq CDf^n(x)$, completing the induction.

With these lemmas, we may now complete the proof of Theorem 2. Fix any $I \in \mathcal{I}(\Gamma)$, and let x be an endpoint of I.

Choose any $C > 1$ and let δ be the number asserted to exist by Lemma 3. Let B denote the open ball with center x and radius δ. By Lemma 2, $Df^n(x)$ tends to zero as $n \to \infty$. By Lemma 3, Df^n tends uniformly to zero on B.

Choose N so large that $|f^m(B)| < \delta/3$ for all $m > N$. Since $x \in \Gamma$, it is f-recurrent. Hence we may choose $k > N$ so that $|f^k(x) - x| < \delta/3$. By choice of N, this means that $f^k(B)$ is strictly contained in B. Therefore f^k has a fixed point in B. This means f has a periodic point, contradicting our hypothesis that f is a Denjoy counterexample.

4. Theorem 1 is sharp

The construction of Denjoy counterexamples is standard (see e.g. Herman [8]). It turns out that the standard construction of a $C^{1+\alpha}$ counterexample does the trick: the box dimension of its minimal set is equal to α, and therefore the lower bound on box dimension in the statement of Theorem 1 cannot be increased. The main point is that the box dimension of such minimal sets is easily calculated using the gap sums of section 2.

We sketch here one method of constructing examples that will serve our purposes. Let R denote any irrational rotation of $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. Choose a Hölder exponent $\alpha \in (0,1)$ and set $\beta = 1/\alpha$.

We may choose $M = M(\beta) \in \mathbf{Z}^+$ large enough that $((M+1)/M)^\beta < 5/4$.

For all $n \in \mathbf{Z}$, define

$$l_n = A/(|n| + M)^\beta,$$

where $A > 0$ is chosen so that $\sum l_n = 1$. For each n, let I_n be a closed interval on $\mathbf{S}^1$ of length l_n, and arrange the collection $\{I_n : n \in \mathbf{Z}\}$ so that the intervals are pairwise disjoint and have the same circular ordering as the R-orbit $\{R^n(0) : n \in \mathbf{Z}\}$. (This can always be done for any irrational rotation and any positive bi-infinite sequence $\{l_n\}$ so long as $\sum l_n \leq 1$.)

The complement $C = \mathbf{S}^1 \setminus \bigcup int(I_n)$ is a Cantor set. If f is defined on $\bigcup int(I_n)$ so that $f|_{I_n}$ is an order-preserving homeomorphism onto I_{n+1}, then f will extend by continuity to a homeomorphism of $\mathbf{S}^1$ with minimal set C.

It follows from the definition of l_n that

$$\sum (l_n)^s < \infty \text{ if and only if } s > \alpha.$$

Therefore, from section 2, $BD(C) = \alpha$.

It remains to specify f so that $f \in C^{1+\alpha}$. Let $\phi : [0,1] \to [0,4]$ be a C^∞ bump function such that $\phi^{-1}(0) = [0,1/3] \cup [2/3,1]$ and $\int \phi \, dx = 1$. For each n, let a_n,b_n denote, respectively, the left and right endpoints of I_n.

Define $g : \mathbf{S}^1 \to \mathbf{R}$ by $g(x) = 1$ for $x \in C$, and

$$g(x) = 1 + (\frac{l_{n+1} - l_n}{l_n})\phi(\frac{x - a_n}{l_n})$$

for $x \in I_n$.

It is easy to see that g is well-defined and continuous on $\mathbf{S}^1$. One can check that the choice of $M(\beta)$ in determining the lengths l_n implies that $0 < g < 2$ on $\mathbf{S}^1$. Also, a simple calculation shows that

$$\int_{a_n}^{b_n} g = l_{n+1} \text{ and } \int_0^1 g = 1.$$

Now define

$$f(x) = a_1 + \int_{a_0}^{x} g(t) \, dt.$$

Since g is positive, continuous, and $\int_0^1 g = 1$, f is (the lift of) a C^1 diffeomorphism of S^1. The reader can verify that $f(I_n) = I_{n+1}$ for all n. The modulus of continuity of $Df = g$ is determined by the estimate on each interval. On I_n, the variation of g is on the order of

$$|\frac{l_{n+1} - l_n}{l_n}| = O(1/|n|)$$

over an interval whose length is $O(1/|n|^\beta)$. Hence g is $(1/\beta = \alpha)$-Hölder and so $f \in C^{1+\alpha}$.

References

1. S. Bates and A. Norton, *On sets of critical values in the real line*, Duke Math. J. **83** (1996), 399–413.
2. A.S. Besicovitch and S.J. Taylor, *On the complementary intervals of a linear closed set of zero Lebesgue measure*, J. London Math. Soc. **29** (1954), 449–459.
3. P. Bohl, *Uber die hinsichtlich der unabhängigen variabeln periodische Differential gleichung erster Ordnung*, Acta Math. **40** (1916), 321–336.
4. A. Denjoy, *Sur les courbes definies par les equations differentielles a la surface du tore*, J. Math Pures et Appl. **11**, 333–375.
5. K.J. Falconer, *Fractal Geometry*, J. Wiley, New York, 1990.
6. G.R. Hall, *Bifurcation of an invariant attracting circle: a Denjoy attractor*, Ergod. Th. & Dynam. Sys. **3** (1983), 87–118.
7. J. Harrison, *Denjoy fractals*, Topology **28** (1989), 59–80.
8. M.R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle a des rotations*, Publ. Math. I.H.E.S. **49** (1979), 5–233.
9. D. McDuff, C^1*-minimal subsets of the circle*, Ann. Inst. Fourier, Grenoble **31** (1981), 177–193.
10. P. McSwiggen, *Diffeomorphisms of the k-torus with wandering domains*, Ergod. Th. & Dynam. Sys. **15** (1995), 1189–1205.
11. A. Norton, *Denjoy minimal sets are far from affine*, to appear, Ergod. Th. & Dynam. Sys.
12. A. Norton and D.P. Sullivan, *Wandering domains and invariant conformal structures for mappings of the 2-torus*, Ann. Acad. Sci. Fenn., Series A I Math. **21** (1996), 51–68.
13. H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle I,II,III,IV*, J. Math. Pures et Appl. (1881,82,85,86).
14. E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.
15. D.P. Sullivan, *Conformal Dynamical Systems*, Geometric Dynamics, Lecture Notes in Math. 1007, Springer-Verlag, New York, 1983, pp. 636–650.

Dept. of Mathematics, University of Texas, Austin, TX 78712
E-mail address: alec@math.utexas.edu