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Periodicities in Linear Fractional Recurrences: Degree growth of birational surface maps

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§0. Introduction

Given complex numbers $\alpha_0, \ldots, \alpha_p$ and $\beta_0, \ldots, \beta_p$, we consider the recurrence relation

$$x_{n+p+1} = \frac{\alpha_0 + \alpha_1 x_{n+1} + \cdots + \alpha_p x_{n+p}}{\beta_0 + \beta_1 x_{n+1} + \cdots + \beta_p x_{n+p}}. \tag{0.1}$$

Thus a p-tuple $(x_1, \ldots, x_p)$ generates an infinite sequence (x_n). We consider two equivalent reformulations in terms of rational mappings: we may consider the mapping $f : \mathbf{C}^p \to \mathbf{C}^p$ given by

$$f(x_1, \ldots, x_p) = \left(x_2, \ldots, x_p, \frac{\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_p x_p}{\beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p} \right). \tag{0.2}$$

Or we may use the imbedding $(x_1, \ldots, x_p) \mapsto [1 : x_1 : \ldots : x_p] \in \mathbf{P}^p$ into projective space and consider the induced map $f : \mathbf{P}^p \to \mathbf{P}^p$ given by

$$f_{\alpha,\beta}[x_0 : x_1 : \ldots : x_p] = [x_0 \beta\cdot x : x_2 \beta\cdot x : \ldots : x_p \beta\cdot x : x_0 \alpha\cdot x], \tag{0.3}$$

where we write $\alpha \cdot x = \alpha_0 x_0 + \cdots + \alpha_p x_p$.

Here we will study the degree growth of the iterates $f^k = f \circ \cdots \circ f$ of f. In particular, we are interested in the quantity

$$\delta(\alpha, \beta) := \lim_{k\to\infty} \left(\mathrm{degree}(f^k_{\alpha,\beta}) \right)^{1/k}.$$

A natural question is: for what values of α and β can (0.1) generate a periodic recurrence? In other words, when does (0.1) generate a periodic sequence (x_n) for all choices of $x_1, \ldots, x_p$? This is equivalent to asking when there is an N such that $f^N_{\alpha,\beta}$ is the identity map. Periodicities in recurrences of the form (0.1) have been studied in [L, KG, KoL, GL, CL]. The question of determining the parameter values α and β for which $f_{\alpha,\beta}$ is periodic has been known for some time and is posed explicitly in [GKP] and [GL, p. 161]. Recent progress in this direction was obtained in [CL]. The connection with our work here is that if $\delta(\alpha,\beta) > 1$, then the degrees of the iterates of $f_{\alpha,\beta}$ grow exponentially, and $f_{\alpha,\beta}$ is far from periodic.

In the case $p = 1$, f is a linear (fractional) map of $\mathbf{P}^1$. The question of periodicity for f is equivalent to determining when a 2×2 matrix is a root of the identity. In this paper we address these questions in the case $p = 2$. In fact, our principal efforts will be devoted to determining $\delta(\alpha,\beta)$ for all of the mappings in the family above. In order to remove trivial cases, we will assume throughout this paper that

$$\begin{gathered} (\alpha_0, \alpha_1, \alpha_2) \text{ is not a multiple of } (\beta_0, \beta_1, \beta_2), \\ (\alpha_1, \beta_1) \neq (0,0), \quad (\alpha_2, \beta_2) \neq (0,0), \text{ and} \\ (\beta_1, \beta_2) \neq (0,0). \end{gathered} \tag{0.4}$$

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Note that if the first condition in (0.4) is not satisfied, then the right hand side of (0.1) is constant. If the left hand part of the second condition (0.4) is not satisfied, then f does not depend on x_1 thus has rank 1, which cannot be periodic. If the right hand part of the second condition (0.4) is not satisfied, then f^2 is essentially the 1-dimensional mapping $\zeta \mapsto \frac{\alpha_0+\alpha_1\zeta}{\beta_0+\beta_1\zeta}$. If the third condition in (0.4) is not satisfied, then f is linear. In this case, the periodicity of f is a question of linear algebra.

Since we consider all parameters satisfying (0.4), we must treat a number of separate cases. Let V_* be the set of (α,β) such that $\beta_1\beta_2 \neq 0$ and $f^3_{\alpha,\beta}\Sigma_\beta = p$, and let V_n denote the variety of parameters (α,β) such that

$$\begin{aligned} &\beta_2 = 0, \text{ and } \ f^n_{\alpha,\beta}(q) = p, \\ \text{where } \ &p = [\beta_1\alpha_2 - \beta_2\alpha_1 : -\beta_0\alpha_2 + \alpha_0\beta_2 : \alpha_1\beta_0 - \alpha_0\beta_1], \\ \text{and } \ &q = [\beta_1(\beta_1\alpha_2) : \beta_1(\alpha_1\beta_0 - \alpha_0\beta_1) : \alpha_1(\beta_1\alpha_2 - \alpha_1\beta_2)]. \end{aligned} \tag{0.5}$$

The following two numbers are of special importance here:

$$\phi \ (\approx 1.61803 \text{ golden mean}) \text{ is the largest root of } x^2 - x - 1 \tag{0.6}$$

$$\delta_\star \ (\approx 1.32472) \text{ is the largest root of } x^3 - x - 1 \tag{0.7}$$

Theorem 1. *Suppose that $(\alpha,\beta) \notin \bigcup_{n\geq 0} V_n$. Then $f_{\alpha,\beta}$ is not birationally conjugate to an automorphism. If $(\alpha,\beta) \in V_*$, then the degree of $f^n_{\alpha,\beta}$ grows linearly in n. Otherwise, $\phi \geq \delta(\alpha,\beta) \geq \delta_\star > 1$. For generic (α,β), the dynamic degree is $\delta(\alpha,\beta) = \phi$.*

In particular, we see that $f_{\alpha,\beta}$ has exponential degree growth in almost all of these cases. The remaining possibilities are:

Theorem 2. *If $(\alpha,\beta) \in V_n$ for some $n \geq 0$, then there is a complex manifold $X = X_{\alpha,\beta}$ obtained by blowing up $\mathbf{P}^2$ at finitely many points, and $f_{\alpha,\beta}$ induces a biholomorphic map $f_{\alpha,\beta} : X \to X$. Further:*

If $n = 0$, $f_{\alpha,\beta}$ is periodic of period 6.
If $n = 1$, $f_{\alpha,\beta}$ is periodic of period 5.
If $n = 2$, $f_{\alpha,\beta}$ is periodic of period 8.
If $n = 3$, $f_{\alpha,\beta}$ is periodic of period 12.
If $n = 4$, $f_{\alpha,\beta}$ is periodic of period 18.
If $n = 5$, $f_{\alpha,\beta}$ is periodic of period 30.
If $n = 6$, the degree of $f^n_{\alpha,\beta}$ is asymptotically quadratic in n.
If $n \geq 7$, $f_{\alpha,\beta}$ has exponential degree growth rate $\delta(\alpha,\beta) = \delta_n > 1$, which is given by the largest root of the polynomial $x^{n+1}(x^3 - x - 1) + x^3 + x^2 - 1$. Further, δ_n increases to $\delta_\star$ as $n \to \infty$.

The family of maps

$$(x, y) \mapsto (y, \frac{a + y}{x})$$

has been studied by several authors (cf. [L, KoL, KLR, GBM, CL]). Within this family, the case $a = 0$ corresponds to V_0, $a = 1$ corresponds to V_1, and all the rest belong to the case V_6 (see §6).

In the cases $n \geq 7$, the entropy of $f_{\alpha,\beta}$ is equal to $\log \delta_n$ by Cantat [C]. The number $\delta_\star$ is known (see [BDGPS, Chap. 7]) to be the infimum of all Pisot numbers. Diller and Favre [DF] showed that if g is a birational surface map which is not birationally conjugate to a holomorphic automorphism, then $\delta(g)$ is a Pisot number. So the maps f in the cases $n \geq 7$ have smaller degree growth than any such g. Note that projective surfaces which have automorphisms of positive entropy are relatively rare: Cantat [C] shows that, except for nonminimal rational surfaces (like X in Theorem 2), the only possibilities are complex tori, K3 surfaces, or Enriques surfaces.

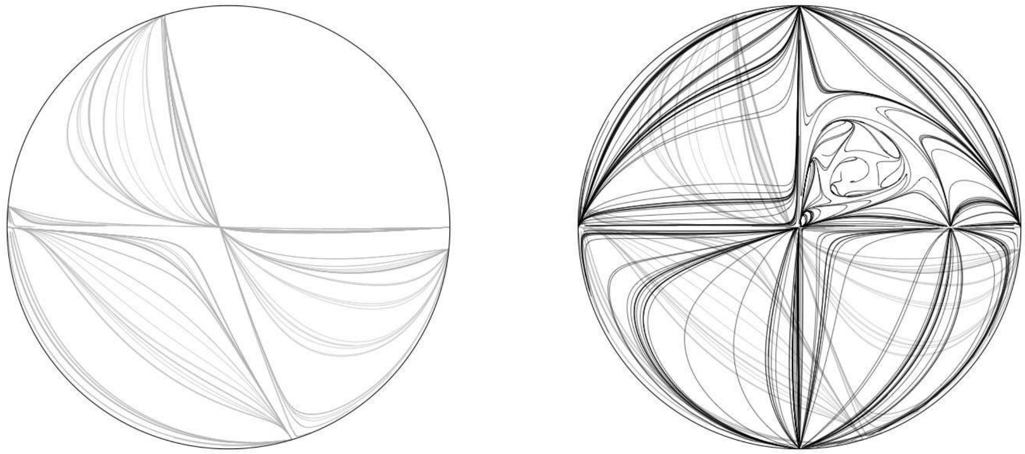


Figure 0.1. A map with (maximal) degree growth ϕ.

Determining the dynamical degree for this family of mappings may be seen as a first step towards the dynamical study of these maps. Figure 0.1 portrays stable and unstable manifolds of a mapping of maximal degree growth within the family $f_{\alpha,\beta}$. This paper is organized as follows. In §1 we give the general properties of the family $f_{\alpha,\beta}$. In §2 we show that $\delta(f_{\alpha,\beta}) = \phi$ if $f_{\alpha,\beta}$ has only two exceptional curves. Next we determine $\delta(f_{\alpha,\beta})$ in the (generic) case where it has three exceptional curves. This determination, however, threatens to involve a large case-by-case analysis. We avoid this by adopting a more general approach. In §3 we show how $\delta(f_{\alpha,\beta})$ may be derived from the set of numbers in open and closed orbit lists. We do this by showing that results of [BK] may be extended from the "elementary" case to the general case. We use this in §4 to determine $\delta(\alpha,\beta)$ when the critical triangle is nondegenerate. In §5 we handle the periodic cases in Theorem 2. In §6, we discuss parameter space and the varieties V_n for $0 \le n \le 6$. We explain the computer pictures in the Appendix.

We wish to thank Curt McMullen and the referee for helpful comments on this paper, and Takato Uehara for finding an error in an earlier version of Theorem 1; and we would like to draw the reader's attention to a recent preprint "Dynamics of blowups of the projective plane," by McMullen, which is available at www.math.harvard.edu/∼ctm.

§1. Setting and Basic Properties

In this section we review some basic properties of the map

$$f(x) = [x_0\beta\cdot x : x_2\beta\cdot x : x_0\alpha\cdot x],$$

which is the map (0.3) in the case $p = 2$. (We refer to [GBM] for a description of f as a real map.) The indeterminacy locus is

$$\begin{aligned}\mathcal{I} =& \{x \in \mathbf{P}^2 : x_0(\beta\cdot x) = x_2(\beta\cdot x) = x_0(\alpha\cdot x) = 0\}\\ =& \{e_1, p_0, p_\gamma\},\end{aligned}$$

where we set $e_1 = [0 : 1 : 0]$, $p_0 = [0 : -\beta_2 : \beta_1]$ and $p_\gamma = [\beta_1\alpha_2 - \beta_2\alpha_1 : -\beta_0\alpha_2 + \alpha_0\beta_2 : \alpha_1\beta_0 - \alpha_0\beta_1]$. Thus f is holomophic on $\mathbf{P}^2 - \mathcal{I}$, and its Jacobian is $2x_0(\beta\cdot x)[\beta_1(\alpha\cdot x) - \alpha_1(\beta\cdot x)]$. Let us set

$$\gamma = (\beta_1\alpha_0 - \alpha_1\beta_0, 0, \beta_1\alpha_2 - \alpha_1\beta_2) \in \mathbf{C}^3$$

and note that the Jacobian vanishes on the curves

$$\Sigma_0 = \{x_0 = 0\}, \quad \Sigma_\beta = \{\beta \cdot x = 0\}, \quad \text{and} \quad \Sigma_\gamma = \{\gamma \cdot x = 0\}.$$

These curves are exceptional in the sense that they are mapped to points:

$$f(\Sigma_0 - \mathcal{I}) = e_1, \quad f(\Sigma_\beta - \mathcal{I}) = e_2 := [0:0:1], \quad f(\Sigma_\gamma - \mathcal{I}) = q, \tag{1.1}$$

where q is defined in (0.5). We write the set of exceptional curves as $\mathcal{E}(f) = \{\Sigma_0, \Sigma_\beta, \Sigma_\gamma\}$.

Lemma 1.1.

$$f(\mathbf{P}^2 - \Sigma_0 \cup \Sigma_\beta) \cap \Sigma_0 = \emptyset.$$

Further, if $\beta_2 \neq 0$,

$$f(\mathbf{P}^2 - \Sigma_0 \cup \{p_\gamma\}) \cap \{p_0\} = \emptyset.$$

Proof. In $\mathbf{P}^2 - \mathcal{E}(f) \cup \mathcal{I}(f)$, f is holomorphic. It follows that for $[x_0 : x_1 : x_2] \in \mathbf{P}^2 - \mathcal{E}(f) \cup \mathcal{I}(f)$, $f([x_0 : x_1 : x_2]) \notin \Sigma_0$ since $x_0(\beta \cdot x) \neq 0$. If $\beta_1 = 0$ or $\beta_1\alpha_2 - \alpha_1\beta_2 = 0$ then either $\Sigma_\gamma = \Sigma_\beta$ or $\Sigma_\gamma = \Sigma_0$. If both β_1 and $\beta_1\alpha_2 - \alpha_1\beta_2$ are non-zero, we have $f(\Sigma_\gamma) = q \notin \Sigma_0$. In case $\beta_2 \neq 0$, for $[x_0 : x_1 : x_2] \in \Sigma_\beta$, we have seen that $f([x_0 : x_1 : x_2]) = e_2 \neq p_0$, which completes the proof.

The inverse of f is given by the map

$$f^{-1}(x) = [x_0 B \cdot x : x_0 A \cdot x - \beta_2 x_1 x_2 : x_1 B \cdot x],$$

where we set $A = (\alpha_0, \alpha_2, -\beta_0)$ and $B = (-\alpha_1, 0, \beta_1)$. In the special case $\beta_2 = 0$, the form of f^{-1} is similar to that of f. The indeterminacy locus $\mathcal{I}(f^{-1}) = \{e_1, e_2, q\}$ consists of the three points which are the f-images of the exceptional lines for f. The Jacobian of f^{-1} is

$$-2x_0(\alpha_1\beta_0 x_0 - \alpha_0\beta_1 x_0 - \alpha_2\beta_1 x_1 + \alpha_1\beta_2 x_1) B \cdot x.$$

Let us set $C = (\alpha_1\beta_0 - \alpha_0\beta_1, \alpha_1\beta_2 - \alpha_2\beta_1, 0)$, and $\Sigma_B = \{x \cdot B = 0\}$, $\Sigma_C = \{x \cdot C = 0\}$. In fact, $\mathcal{E}(f^{-1}) = \{\Sigma_0, \Sigma_B, \Sigma_C\}$, and f^{-1} acts as: $\Sigma_0 \mapsto p_0$, $\Sigma_B \mapsto e_1$, and $\Sigma_C \mapsto p_\gamma$.

To understand the behavior of f at $\mathcal{I}$, we define the cluster set $Cl_f(a)$ of a point $a \in \mathbf{P}^2$ by

$$Cl_f(a) = \{x \in \mathbf{P}^2 : x = \lim_{a' \to a} f(a'),\ a' \in \mathbf{P}^2 - \mathcal{I}(f)\}.$$

In general, a cluster set is connected and compact. In our case, we see that the cluster set is a single point when $a \notin \mathcal{I}$, i.e., when f is holomorpic. And the cluster sets of the points of indeterminacy are found by applying f^{-1}: i.e., $e_1 \mapsto Cl_f(e_1) = \Sigma_B$, $p_0 \mapsto Cl_f(p_0) = \Sigma_0$, and $p_\gamma \mapsto Cl_f(p_\gamma) = \Sigma_C$. Thus f acts as in Figure 1.1: the lines on the left hand triangle are exceptional and are mapped to the vertices of the right hand triangle, and the vertices of the left hand triangle are blown up to the sides of the right hand triangle.

Let

$$\pi : Y \to \mathbf{P}^2 \tag{1.2}$$

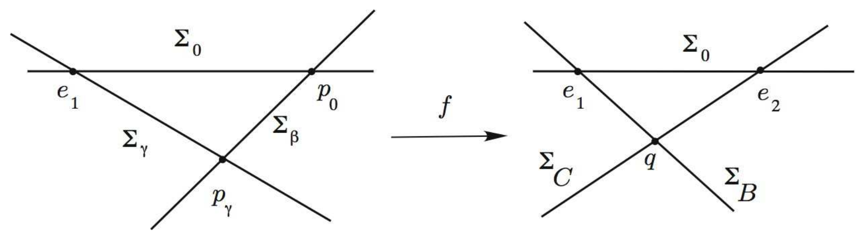
be the complex manifold obtained by blowing up $\mathbf{P}^2$ at e_1. We will discuss the induced birational map $f_Y : Y \to Y$. We let $E_1 := \pi^{-1}e_1$ denote the exceptional blow-up fiber. The projection gives a biholomorphic map $\pi : Y - E_1 \to \mathbf{P}^2 - e_1$. For a complex curve $\Gamma \subset \mathbf{P}^2$, we use the notation $\Gamma \subset Y$ to denote the *strict transform* of Γ in Y. Namely, Γ denotes the closure of $\pi^{-1}(\Gamma - e_1)$ inside Y. Thus Γ is a proper subset of $\pi^{-1}\Gamma = \Gamma \cup E_1$.

We identify E_1 with $\mathbf{P}^1$ in the following way. For $[\xi_0 : \xi_2] \in \mathbf{P}^1$, we associate the point

$$[\xi_0 : \xi_2]_{E_1} := \lim_{t\to 0} \pi^{-1}[t\xi_0 : 1 : t\xi_2] \in E_1.$$

We may now determine the map f_Y on Σ_0. For $x = [0 : x_1 : x_2] = \lim_{t\to 0}[t : x_1 : x_2] \in \Sigma_0$, we assign $f_Y x := \lim_{t\to 0} f[t : x_1 : x_2] \in Y$. That is, $f[t : x_1 : x_2] = [t\beta \cdot x : x_2\beta \cdot x : t\alpha \cdot x]$, and so taking the limit as $t \to 0$, we obtain

$$f_Y[0 : x_1 : x_2] = [\beta \cdot x : \alpha \cdot x]_{E_1}. \tag{1.3}$$



$p_0 \mapsto \Sigma_0 \mapsto e_1 \mapsto \Sigma_B \qquad \Sigma_\beta \mapsto e_2 \qquad \Sigma_\gamma \mapsto q \qquad p_\gamma \mapsto \Sigma_C$

Figure 1.1. Blowing-up/blowing-down behavior of f.

Now we make a similar computation for a point $[\xi_0 : \xi_2]_{E_1}$ in the fiber E_1 over the point of indeterminacy e_1. We set $x = [t\xi_0 : 1 : t\xi_2]$ so that

$$fx = [t\xi_0\beta \cdot x : t\xi_2\beta \cdot x : t\xi_0\alpha \cdot x].$$

Taking the limit as $t \to 0$, we find

$$f_Y([\xi_0 : \xi_2]_{E_1}) = [\xi_0\beta_1 : \xi_2\beta_1 : \xi_0\alpha_1] \in \Sigma_A.$$

Thus we have:

Lemma 1.2. *The map f_Y has the properties:*
(i) f_Y is a local diffeomorphism at points of Σ_0 if and only if $\beta_1\alpha_2 - \alpha_1\beta_2 \neq 0$.
(ii) f_Y is a local diffeomorphism at points of E_1 if and only if $\beta_1 \neq 0$.

§2. Degenerate Critical Triangle

We will refer to the set $\{\Sigma_0, \Sigma_\beta, \Sigma_\gamma\}$ of exceptional curves as the *critical triangle*; we say that the critical triangle is *nondegenerate* if these three curves are distinct. Since $(\beta_1, \beta_2) \neq (0, 0)$, we have $\Sigma_0 \neq \Sigma_\beta$. Thus there are only two possibilities for a degenerate triangle. The first of these is the case $\Sigma_\gamma = \Sigma_\beta$, which occurs when $\beta_1 = 0$. The second is $\Sigma_\gamma = \Sigma_0$, which occurs when $\beta_1\alpha_2 - \alpha_1\beta_2 = 0$. (And since $\Sigma_0 \neq \Sigma_\beta$ we have $\beta_1 \neq 0$ in this case.) We will show that $\delta(\alpha, \beta) = \phi$ when the critical triangle is degenerate. This is different from the general case (and easier), and we treat it in this section.

In order to determine the degree growth rate of f, we will consider the induced pullback f^* on $H^{1,1}$. We will be working on compact, complex surfaces X for which $H^{1,1}(X)$ is generated by the classes of divisors. If $[D]$ is the divisor of a curve $D \subset X$, then we define $f^*[D]$ to be the class of the divisor $f^{-1}D$. We say that f is *1-regular* if $(f^n)^* = (f^*)^n$ for all $n \geq 0$. Fornaess and Sibony showed in [FS] that if

$$\text{for every exceptional curve } C \text{ and all } n \geq 0, f^n C \notin \mathcal{I} \tag{2.1}$$

then f is 1-regular. We will use this criterion in the following:

Proposition 2.1. *If the critical triangle is degenerate, then the map $f_Y : Y \to Y$ is 1-regular.*

Proof. We treat the two possibilities separately. The first case is $\Sigma_\gamma = \Sigma_\beta$; see Figure 2.1. In this case f has two exceptional lines Σ_0 and Σ_β and two points of indeterminacy $\mathcal{I} = \{e_1, p_\lambda\}$. After we blow up e_1 to obtain Y, the line Σ_0 is no longer exceptional. (Our drawing convention in this and subsequent Figures is that exceptional curves are thick, and points of indeterminacy are circled.) By (1.3), we see that f_Y maps E_1 to $e_2 = q$, and thus the exceptional set becomes $\mathcal{E}(f_Y) = \{E_1, \Sigma_\beta = \Sigma_\gamma\}$. Now in order to check condition (2.1), we need to follow the orbit of e_2. By (1.3) we see that e_2 is part of a 2-cycle $\{e_2, [\beta_2 : \alpha_2]_{E_1}\}$. On the other hand, the points of indeterminacy for f_Y are p_γ and $[0:1]_{E_1} = E_1 \cap \Sigma_0$. Since $\beta_1 = 0$ in this case, we have $\beta_2 \neq 0$, so (2.1) holds.

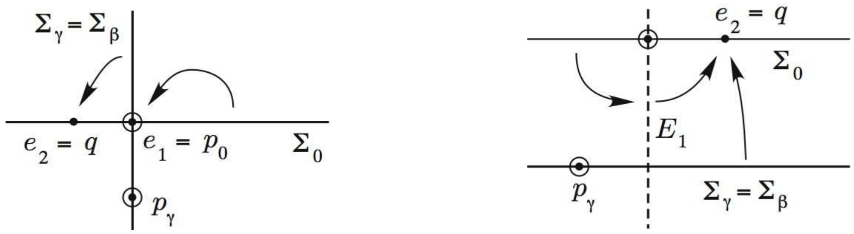


Figure 2.1. The case $\Sigma_\beta = \Sigma_\gamma$.

The second case is $\Sigma_\gamma = \Sigma_0$. Again, $\mathcal{I} = \{e_1, p_\gamma\}$, but $\mathcal{E}(f) = \{\Sigma_0, \Sigma_\beta\}$, and the arrangement of exceptional curves and points of indeterminacy are as in Figure 2.2. In this case, we have $\beta_1 \neq 0$, so by Lemma 1.2, we have $\mathcal{I}(f_Y) = \{p_0 = p_\gamma\}$ and $\mathcal{E}(f_Y) = \{\Sigma_\beta\}$. As before, we need to track the orbit of e_2. But by Lemma 1.1, we see that we can never have $f^j e_2 = p_0$ for $j \geq 1$. Thus (2.1) holds in this case, too, and the proof is complete.

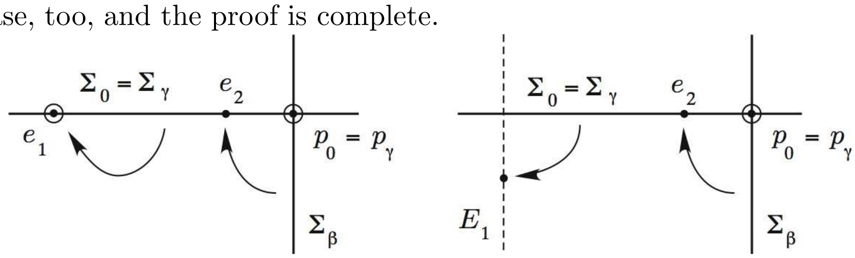


Figure 2.2. The case $\Sigma_0 = \Sigma_\gamma$.

Now let us determine f_Y^*. The cohomology group $H^{1,1}(\mathbf{P}^2; \mathbf{Z})$ is one-dimensional and is generated by the class of a complex line. We denote this generator by L. Let $L_Y := \pi^* L \in H^{1,1}(Y; \mathbf{Z})$ be the class induced by the map (1.2). It follows that $\{L_Y, E_1\}$ is a basis for $H^{1,1}(Y; \mathbf{Z})$. Now $\Sigma_0 = L \in H^{1,1}(\mathbf{P}^2; \mathbf{Z})$. Pulling this back by π, we have

$$L_Y = \pi^* \Sigma_0 = \Sigma_0 + E_1.$$

Now f_Y^* acts by taking pre-images:

$$f_Y^* E_1 = [f^{-1} E_1] = \Sigma_0 = L_Y - E_1,$$

where the last equality follows from the equation above.

Now e_1 is indeterminate, and $f e_1 = \Sigma_A$. Since Σ_A intersects any line L, it follows that $e_1 \in f^{-1} L$. Thus

$$\pi^*[f^{-1}L] = [f^{-1}L] + E_1 \in H^{1,1}(Y; \mathbf{Z}).$$

On the other hand, $f^{-1}L = 2L \in H^{1,1}(\mathbf{P}^2; \mathbf{Z})$. Thus

$$\pi^*[f^{-1}L] = \pi^* 2L = 2L_Y.$$

Putting these last two equations together, we have $f_Y^* L_Y = 2L_Y - E_1$. Thus

$$f_Y^* = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix},$$

which is a matrix with spectral radius equal to ϕ. This yields the following:

Proposition 2.2. *If the critical triangle is degenerate, then $\delta(\alpha,\beta) = \phi$.*

§3. Regularization and Degree Growth

In this Section we discuss a different, but more general, family of maps. By $J : \mathbf{P}^2 \to \mathbf{P}^2$ we denote the involution

$$J[x_0 : x_1 : x_2] = [x_0^{-1} : x_1^{-1} : x_2^{-1}] = [x_1x_2 : x_0x_2 : x_0x_1].$$

For an invertible linear map L of $\mathbf{P}^2$ we consider the map $f := L \circ J$. The exceptional curves are $\mathcal{E} = \{\mathbf{\Sigma}_0, \mathbf{\Sigma}_1, \mathbf{\Sigma}_2\}$, where $\mathbf{\Sigma}_j := \{x_j = 0\}, j = 0, 1, 2$, and the points of indeterminacy are $\mathcal{I} = \{\epsilon_0, \epsilon_1, \epsilon_2\}$, where $\epsilon_i = \mathbf{\Sigma}_j \cap \mathbf{\Sigma}_k$, with $\{i, j, k\} = \{0, 1, 2\}$. We define $\mathbf{a}_j := f(\mathbf{\Sigma}_j - \mathcal{I}) = L\epsilon_j$ for $j = 0, 1, 2$.

For $p \in \mathbf{P}^2$ we define the *orbit* $\mathcal{O}(p)$ as follows. If $p \in \mathcal{E} \cup \mathcal{I}$, then $\mathcal{O}(p) = \{p\}$. If there exists an $N \geq 1$ such that $f^j p \notin \mathcal{E} \cup \mathcal{I}$ for $0 \leq j \leq N-1$ and $f^N p \in \mathcal{E} \cup \mathcal{I}$, then we set $\mathcal{O}(p) = \{p, fp, \dots, f^N p\}$. Otherwise we have $f^j p \notin \mathcal{E} \cup \mathcal{I}$ for all $j \geq 0$, and we set $\mathcal{O}(p) = \{p, fp, f^2p, \dots\}$. We say the orbit $\mathcal{O}(p)$ is *singular* if it is finite; otherwise, it is non-singular. We say an orbit $\mathcal{O}(p)$ is *elementary* if it is either non-singular, or if it ends at a point of indeterminacy. In other words, a non-elementary orbit ends in a point of $\mathcal{E} - \mathcal{I}$. We say that f is elementary if all of its singular orbits are elementary.

Let us write $\mathcal{O}_i = \mathcal{O}(\mathbf{a}_i) = \mathcal{O}(f(\mathbf{\Sigma}_i - \mathcal{I}))$ for the orbit of an exceptional curve. We set

$$S = \{i \in \{0, 1, 2\} : \mathcal{O}_i \text{ is singular}\},$$

and

$$S_0 = \{i \in \{0, 1, 2\} : \mathcal{O}_i \text{ is singular and elementary}\}.$$

Lemma 3.1. *If f is not 1-regular, then it has a singular orbit that is elementary. Thus $S_0 \neq \emptyset$.*

Proof. Suppose for all $i \in S_0$, $\mathcal{O}_i$ is non-singular. If follows that every orbit $\mathcal{O}_j, j \notin S_0$ ends at a point in $\mathbf{\Sigma}_i, i \in S_0$. Since all $\mathcal{O}_i, i \in S_0$ are non-singular, $\mathbf{\Sigma}_j, j \notin S_0$ cannot end at a point of indeterminacy. This means that f is 1-regular.

Henceforth, we will assume that f is not 1-regular. Let $\mathcal{O}_{S_0} = \bigcup_{i \in S_0} \mathcal{O}_i$. We write $X_0 = \mathbf{P}^2$, and let $\pi : X_1 \to X_0$ be the complex manifold obtained by blowing up the points of $\mathcal{O}_{S_0}$. We let $f_1 : X_1 \to X_1$ denote the induced birational mapping. By Lemma 1.2, we see that the curves $\mathbf{\Sigma}_i$, $i \in S_0$, are not exceptional for f_1, and the blowing up operation constructed no new points of indeterminacy for f_1. Thus the exceptional curves for f_1 are $\mathbf{\Sigma}_i$ for $i \notin S_0$. If S_0 is a proper subset of S, then for $i \in S - S_0$ we redefine $\mathcal{O}_i$ to be the f_1-orbit of $\mathbf{a}_i$ inside X_1. Let us define $S_1 = \{i \in S - S_0 : \mathcal{O}_i \text{ is elementary for } f_1\}$. We may apply Lemma 3.1 to conclude that if $S - S_0 \neq \emptyset$, and f_1 is not regular, then $S_1 \neq \emptyset$. As before, we may define $\mathcal{O}_{S_1} = \bigcup_{i \in S_1} \mathcal{O}_i$, and we construct the complex manifold $\pi : X_2 \to X_1$ by blowing up all the points of $\mathcal{O}_{S_1}$. Doing this, we reach the situation where f_k is 1-regular for some $1 \leq k \leq 3$, and for each $i \in \bigcup_{j=0}^{k-1} S_j$, the orbit $\mathcal{O}_i$ has the form $\mathcal{O}_i = \{\mathbf{a}_i, \dots, \epsilon_{\tau(i)}\}$ for some $\tau(i) \in \{0, 1, 2\}$.

Next we work with the map f_k and organize the singular orbits $\mathcal{O}_i$ into lists. Two orbits $\mathcal{O}_1 = \{\mathbf{a}_1, \dots, \epsilon_{j_1}\}$ and $\mathcal{O}_2 = \{\mathbf{a}_2, \dots, \epsilon_{j_2}\}$ are in the same list if either $j_1 = 2$ or $j_2 = 1$, i.e., if the ending index of one orbit is the same as the beginning index of the other. (This definition is given in more detail in [BK, §4].) In our case, the possible lists are as follows (modulo permutation of the indices $\{0,1,2\}$). If there is only one singular orbit, we have the list $\mathcal{L} = \{\mathcal{O}_i = \{\mathbf{a}_i, \dots, \epsilon_{\tau(i)}\}\}$. If $\tau(i) = i$, we say that $\mathcal{L}$ is a *closed list*; otherwise it is an *open list*. If there are two singular orbits, we can have two closed lists:

$$\mathcal{L}_1 = \{\mathcal{O}_0 = \{\mathbf{a}_0, \dots, \epsilon_0\}\},\ \mathcal{L}_2 = \{\mathcal{O}_1 = \{\mathbf{a}_1, \dots, \epsilon_1\}\}$$

or a closed list and an open list:

$$\mathcal{L}_1 = \{\mathcal{O}_0 = \{\mathbf{a}_0, \dots, \epsilon_0\}\},\ \mathcal{L}_2 = \{\mathcal{O}_1 = \{\mathbf{a}_1, \dots, \epsilon_2\}\}.$$

We cannot have two open lists since there are only 3 orbits $\mathcal{O}_i$. We can also have a single list :

$$\mathcal{L} = \{\mathcal{O}_0 = \{\mathbf{a}_0, \dots, \epsilon_1\}, \mathcal{O}_1 = \{\mathbf{a}_1, \dots, \epsilon_{\tau(1)}\}\},$$

which is a closed list if $\tau(1) = 0$ and an open list otherwise. If there are three singular orbits, then the possibilities are

$$\mathcal{L} = \{\mathcal{O}_0 = \{\mathbf{a}_0, \dots, \epsilon_1\}, \mathcal{O}_1 = \{\mathbf{a}_1, \dots, \epsilon_2\}, \mathcal{O}_2 = \{\mathbf{a}_2, \dots, \epsilon_0\}\},$$

$$\mathcal{L}_1 = \{\mathcal{O}_0 = \{\mathbf{a}_0, \dots, \epsilon_0\}\},\ \mathcal{L}_2 = \{\mathcal{O}_1 = \{\mathbf{a}_1, \dots, \epsilon_2\}, \mathcal{O}_2 = \{\mathbf{a}_2, \dots, \epsilon_1\}\},$$

or

$$\mathcal{L}_1 = \{\mathcal{O}_0 = \{\mathbf{a}_0, \dots, \epsilon_0\}\}, \mathcal{L}_2 = \{\mathcal{O}_1 = \{\mathbf{a}_1, \dots, \epsilon_1\}\}, \mathcal{L}_3 = \{\mathcal{O}_2 = \{\mathbf{a}_2, \dots, \epsilon_2\}\},$$

where all the lists are closed.

For an orbit $\mathcal{O}_i$, we let $n_i = |\mathcal{O}_i|$ denote its length, and for an orbit list $\mathcal{L} = \{\mathcal{O}_a, \dots, \mathcal{O}_{a+\mu}\}$, we denote the set of orbit lengths by $|\mathcal{L}| = \{n_a, \dots, n_{a+\mu}\}$. We set $\#\mathcal{L}^c = \{|\mathcal{L}_j| : \mathcal{L}_j \text{ is closed}\}$ and $\#\mathcal{L}^o = \{|\mathcal{L}_j| : \mathcal{L}_j \text{ is open}\}$. The set $\#\mathcal{L}^c$ and $\#\mathcal{L}^o$ determine $\delta(f)$, as is shown in the following:

Lemma 3.2. *The orbit structures $\#\mathcal{L}^c$, $\#\mathcal{L}^o$ determine f_k^* up to conjugacy.*

Proof. First let us suppose that f is elementary and show how to determine f_1^* from $\#\mathcal{L}^c$ and $\#\mathcal{L}^o$. In this case we have $S = S_0$. We set $X := X_1$. It follows from (2.1) that $f_X : X \to X$ is 1-regular. For $p \in \mathcal{O}_S - \mathcal{I}$ we let $\mathcal{F}_p = \pi^{-1}p$ denote the exceptional fiber over p. If $\epsilon_i \in \mathcal{O}_S \cap \mathcal{I}$, we let E_i denote the exceptional fiber over ϵ_i. We will feel free to identify curves with the classes they generate in $H^{1,1}(X)$. Let $H \in H^{1,1}(\mathbf{P}^2)$ denote the class of a line, and let $H_X = \pi^* H$ denote the induced class in $H^{1,1}(X)$. For $i \in S$, we have

$$\mathbf{\Sigma}_i \to \mathbf{a}_i \to \cdots \to f^{n_i - 1}\mathbf{a}_i = f^{n_i}(\mathbf{\Sigma}_i - \mathcal{I}) = \epsilon_{\tau(i)}$$

for some $\tau(i) \in \{0,1,2\}$. At each point $f^j \mathbf{a}_i$, $0 \le j \le n_i - 1$, f is locally biholomorphic, so f_X induces a biholomorphic map

$$f_X : \mathcal{F}_{f^j \mathbf{a}_i} \to \mathcal{F}_{f^{j+1}\mathbf{a}_i} \quad 0 \le j \le n_i - 2, \text{ and}$$

$$f_X : \mathcal{F}_{f^{n_i - 1}\mathbf{a}_i} \to E_{\tau(i)}.$$

It follows that

$$\begin{aligned} &f_X^* \mathcal{F}_{f^{j+1}\mathbf{a}_i} = \mathcal{F}_{f^j \mathbf{a}_i} \quad \text{for } 0 \le j \le n_i - 2,\ i \in S \\ &f_X^* E_{\tau(i)} = \mathcal{F}_{f^{n_i - 1}\mathbf{a}_i} \end{aligned} \tag{3.1}$$

and
$$f_X^*\mathcal{F}_{\mathbf{a}_i} = \{\mathbf{\Sigma}_i\} \quad \text{for } i \in S \tag{3.2}$$
where $\{\mathbf{\Sigma}_i\}$ is the class induced by $\mathbf{\Sigma}_i$ in $H^{1,1}(X)$. Let $\Omega = \mathcal{I} \cap \{\epsilon_{\tau(i)} = f^{n_i-1}\mathbf{a}_i,\ i \in S\}$, the set of blow-up centers which belong to $\mathcal{I}$. Let $\mathcal{A}$ denote the set of indices i such that $\mathcal{O}_i$ is a singular orbit and is the first orbit in an open orbit list. For each i, $\mathbf{\Sigma}_i$ contains blow-up centers in the set $\Omega - \{\epsilon_i\}$. Notice that if $i \in \mathcal{A}$, $\epsilon_i \notin \Omega$, otherwise $\epsilon_i \in \Omega$. Using the identity $\pi^*\{\mathbf{\Sigma}_i\} = \{\pi^{-1}\mathbf{\Sigma}_i\}$, we have
$$\begin{aligned} \{\mathbf{\Sigma}_i\} &= H_X - E_\Omega + E_i \quad i \notin \mathcal{A} \\ \{\mathbf{\Sigma}_i\} &= H_X - E_\Omega \quad i \in \mathcal{A} \end{aligned} \tag{3.3}$$
where $E_\Omega := \sum_{\epsilon_t \in \Omega} E_t$. A generic hyperplane $\mathcal{H}$ in $\mathbf{P}^2$ does not contain any blow-up centers and may be considered to be subset of X. Let us restrict the map to $X - \mathcal{I}$. A generic hyperplane $\mathcal{H}$ intersects any line in $\mathbf{P}^2$. It follows that $\epsilon_i \in f_X^{-1}\mathcal{H}$, $i \in \Omega$ and we have
$$2H_X = \pi^*(f^*H) = \pi^*\{f^{-1}\mathcal{H}\} = f_X^*H_X + E_\Omega.$$
Therefore under f_X^*, we have
$$f_X^*H_X = 2H_X - E_\Omega. \tag{3.4}$$
From this, we see that the linear transformation f_X^* is essentially determined by $\#\mathcal{L}^c$ and $\#\mathcal{L}^o$.

Now let us suppose that f and g are two maps with the same orbit list structure, $\#\mathcal{L}^c, \#\mathcal{L}^o$; but f is elementary, and g is not. We have shown that f_1^* is represented by the transformation (3.1–4). Let g_k denote the 1-regularization of g, as given after Lemma 3.1. We claim that under an appropriate choice of basis, the g_k^* will be represented by the same matrix as f_1^*. Rather than carry out the details in general, we illustrate this with an example which appears later in the paper. (The matrix computation for the other cases is similar.) We consider the case where the list structures of f and g are both given by
$$\#\mathcal{L}^o = \emptyset, \quad \#\mathcal{L}^c = \{\{1,6\}\}.$$
For the elementary map f, we may suppose that the singular orbits are $\mathcal{O}_1 = \{\mathbf{a}_1 = \epsilon_2\}$ and $\mathcal{O}_2 = \{\mathbf{a}_2, f\mathbf{a}_2, f^2\mathbf{a}_2, f^3\mathbf{a}_2, f^4\mathbf{a}_2, f^5\mathbf{a}_2 = \epsilon_1\}$ with $f^j\mathbf{a}_2 \notin \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ for $0 \le j \le 4$. We construct $X = X_f$ by blowing up both orbits, and we fix the basis $\mathcal{B}_f = \{H_X, E_2, E_1 = \mathcal{F}_{\epsilon_1}, \mathcal{F}_{f^4\mathbf{a}_2}, \mathcal{F}_{f^3\mathbf{a}_2}, \mathcal{F}_{f^2\mathbf{a}_2}, \mathcal{F}_{f\mathbf{a}_2}, \mathcal{F}_{\mathbf{a}_2}\}$.

If g has the same list structure, we may suppose g has an orbit $\mathcal{O}_1 = \{\mathbf{a}_1 = \epsilon_2\}$, and we construct X_1 by blowing up ϵ_2. Further, we may suppose that $g_1 = g_{X_1}$ has an orbit of the form $\mathcal{O}_2 = \{\mathbf{a}_2, g_1\mathbf{a}_2 \in \mathbf{\Sigma}_1, g_1^2\mathbf{a}_2 \in E_2, g_1^3\mathbf{a}_2, g_1^4\mathbf{a}_2, g_1^5\mathbf{a}_2 = \epsilon_1\}$. We let X_2 be the space obtained from X_1 by blowing up the orbit $\mathcal{O}_2$, and let $g_2 : X_2 \to X_2$ be the induced map. The blowup fibers are E_2, and $\mathcal{F}_{g_1^j\mathbf{a}_2}$, $0 \le j \le 5$. The essential difference between X_f and X_2 is that the exceptional (blowup) fiber $\mathcal{F}_{g_1^2\mathbf{a}_2}$ is created over the blowup fiber E_2. We will use the ordered basis $\mathcal{B}_g = \{H_{X_2}, E_2, E_1 = \mathcal{F}_{\epsilon_1}, \mathcal{F}_{g_1^4\mathbf{a}_2}, \mathcal{F}_{g_1^3\mathbf{a}_2}, \mathcal{F}_{g_1^2\mathbf{a}_2}, \mathcal{F}_{g_1\mathbf{a}_2}, \mathcal{F}_{\mathbf{a}_2}\}$. By (3.1–4), f_X^* is represented with respect to $\mathcal{B}_f$ by the matrix M_1. Now we pass from f^* to g_2^* and show how to go from M_1 to M_2. Since $g_1^2a_2 \in E_2$, it lies over the point of indeterminacy ϵ_2, so we have to add a -1 in the first column. Since $g_1a_2, g_1^2a_2 \in \Sigma_1 = g_1^*E_2$, we have to add two -1's to the second column. Thus g_2^* is represented with respect to $\mathcal{B}_g$ by M_2:
$$M_1 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now define $\hat{\mathcal{B}}_g = \{H_{X_2}, \hat{E}_2 = E_2 + \mathcal{F}_{g_1^2\mathbf{a}_2}, E_1 = \mathcal{F}_{\epsilon_1}, \mathcal{F}_{g_1^4\mathbf{a}_2}, \mathcal{F}_{g_1^3\mathbf{a}_2}, \mathcal{F}_{g_1^2\mathbf{a}_2}, \mathcal{F}_{g_1\mathbf{a}_2}, \mathcal{F}_{\mathbf{a}_2}\}$. We see that with respect to the basis $\hat{\mathcal{B}}_g$, g_2^* is represented by the matrix M_1. Thus M_1 and M_2 have the same characteristic polynomial.

In general, we need to consider an analogous situation, where we blow up a point $p_0 \in X_0$ to produce a fiber $\mathcal{F}_1$. Then we blow up $p_1 \in \mathcal{F}_1$ and produce a fiber $\mathcal{F}_2$, etc., to have a sequence of points $p_j \in \mathcal{F}_j$, $0 \le j \le h-1$ and blow ups fibers $\mathcal{F}_{j+1}$ over p_j. The exceptional fibers $\mathcal{F}_1, \dots, \mathcal{F}_h$ will appear in the basis $\mathcal{B}$. In order to pass to the basis $\hat{\mathcal{B}}$, we replace $\mathcal{F}_j$ by $\hat{\mathcal{F}}_j := \mathcal{F}_j + \mathcal{F}_{j+1} + \cdots + \mathcal{F}_h$.

For each orbit list $\mathcal{L}$, we let $N_{\mathcal{L}} = n_a + \cdots + n_{a+\mu}$ denote the sum of elements of $|\mathcal{L}|$. If $\mathcal{L}$ is closed, we define $T_{\mathcal{L}}(x) = x^{N_{\mathcal{L}}} - 1$, and if $\mathcal{L}$ is open, we define $T_{\mathcal{L}}(x) = x^{N_{\mathcal{L}}}$. We define $S_{\mathcal{L}}$ as:

$$\begin{aligned} S_{\mathcal{L}}(x) &= 1 && \text{if } |\mathcal{L}| = \{n_1\} \\ &= x^{n_1} + x^{n_2} + 2 && \text{if } \mathcal{L} \text{ is closed and } |\mathcal{L}| = \{n_1, n_2\} \\ &= x^{n_1} + x^{n_2} + 1 && \text{if } \mathcal{L} \text{ is open and } |\mathcal{L}| = \{n_1, n_2\} \\ &= \sum_{i=1}^{3} [x^{N_{\mathcal{L}} - n_i} + x^{n_i}] + 3 && \text{if } \mathcal{L} \text{ is closed and } |\mathcal{L}| = \{n_1, n_2, n_3\} \\ &= \sum_{i=1}^{3} x^{N_{\mathcal{L}} - n_i} + \sum_{i \ne 2} x^{n_i} + 1 && \text{if } \mathcal{L} \text{ is open and } |\mathcal{L}| = \{n_1, n_2, n_3\}. \end{aligned}$$

Theorem 3.3. *If $f = L \circ J$, then the dynamic degree $\delta(f)$ is the largest real zero of the polynomial*

$$\chi(x) = (x-2) \prod_{\mathcal{L} \in \mathcal{L}^c \cup \mathcal{L}^o} T_{\mathcal{L}}(x) + (x-1) \sum_{\mathcal{L} \in \mathcal{L}^c \cup \mathcal{L}^o} S_{\mathcal{L}}(x) \prod_{\mathcal{L}' \ne \mathcal{L}} T_{\mathcal{L}'}(x). \tag{3.5}$$

Here $\mathcal{L}$ runs over all orbit lists.

Proof. By Lemma 3.2, we may assume that the orbit list structure belongs to an elementary map. The computation given in the Appendix of [BK] then shows that (3.5) is the characteristic polynomial of f_X^*.

§4. Non-degenerate Critical Triangle

In this section we will determine the degree growth rate of f with non-degenerate critical triangle. As we noted at the beginning of §2, this is equivalent to the condition

$$\beta_1(\beta_1\alpha_2 - \alpha_1\beta_2) \ne 0. \tag{4.1}$$

In particular, the curves Σ_γ, Σ_β and Σ_0 are distinct, as well as $\{e_1, e_2, q\}$, the points of indeterminacy of f^{-1}. Let us choose invertible linear maps M_1 and M_2 of $\mathbf{P}^2$ such that

$$M_1\mathbf{\Sigma}_0 = \Sigma_0,\ M_1\mathbf{\Sigma}_1 = \Sigma_\beta,\ M_1\mathbf{\Sigma}_2 = \Sigma_\gamma,$$

and

$$M_2 e_1 = \epsilon_0,\ M_2 e_2 = \epsilon_1,\ M_2 q = \epsilon_2.$$

It follows that $M_2 \circ f_{\alpha,\beta} \circ M_1$ is a quadratic map with $\mathbf{\Sigma}_j \leftrightarrow e_j$ and so is equal to the map J. Thus $f_{\alpha,\beta}$ is linearly conjugate to a mapping of the form $L \circ J$. When we treat $f_{\alpha,\beta}$ as a mapping $L \circ J$, we make the identifications

$$\mathbf{\Sigma}_0 = \Sigma_0, \quad \mathbf{\Sigma}_1 = \Sigma_\beta, \quad \mathbf{\Sigma}_2 = \Sigma_\gamma,$$

$$\epsilon_0 = p_\gamma, \quad \epsilon_1 = e_1, \quad \epsilon_2 = p_0,$$

and

$$\mathbf{a}_0 = f(\mathbf{\Sigma}_0 - \mathcal{I}(f)) = e_1, \quad \mathbf{a}_1 = f(\mathbf{\Sigma}_1 - \mathcal{I}(f)) = e_2, \quad \mathbf{a}_2 = f(\mathbf{\Sigma}_2 - \mathcal{I}(f)) = q.$$

In all cases, we have $f(\mathbf{\Sigma}_0 - \mathcal{I}) = \mathbf{a}_0 = \epsilon_1$, so the orbit $\mathcal{O}_0 = \{\mathbf{a}_0 = \epsilon_1\}$ is singular and has length one. Let us start with the most exceptional case.

Theorem 4.1. *If $(\alpha, \beta) \in V_*$, then the critical triangle is non-degenerate, and the degree of $f^n_{\alpha,\beta}$ is asymptotically linear in n.*

Proof. Let us claim first that the critical triangle is non-degenerate. Since $\beta_1\beta_2 \neq 0$, Σ_β and Σ_γ are distinct. Thus the only possibility for the triangle to be degenerate is $\Sigma_\gamma = \Sigma_0$. But by Proposition 2.1 we see that $\epsilon_0 = \epsilon_2$ and $f^j(\mathbf{a}_1) \neq \epsilon_2$ for all $j \geq 1$. It follows that $f^{j+1}\mathbf{\Sigma}_1 \neq \epsilon_2$ for all $j \geq 1$. Thus $(\alpha, \beta) \notin V_*$.

Since $\mathbf{\Sigma}_0 \to \epsilon_1$, we blow up ϵ_1 and obtain the space Y as in (1.2). The orbit of $\mathbf{\Sigma}_1$ is now given by

$$f_Y: \quad \mathbf{\Sigma}_1 - \mathcal{I} \to \mathbf{a}_1 \to [\beta_2 : \alpha_2]_{E_1} \to \epsilon_0 \in \mathcal{I}.$$

Let Z be the space obtained by blowing up this orbit in Y. By the second statement in Lemma 1.1, the orbit $\mathcal{O}_2$ is not singular, so f_Z is 1-regular. It follows that with respect to the ordered basis H_Z, E_1, E_0, $\mathcal{F}_{f(\mathbf{a}_1)}$, $\mathcal{F}_{\mathbf{a}_1}$, we have

$$f_Z^* = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are 0 and ± 1, and the canonical form contains a 2×2 Jordan block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so it has linear growth.

If $(\alpha, \beta) \notin V_*$, there are two possibilities for the exceptional component $\mathbf{\Sigma}_1$; the first is that $\mathbf{a}_1 \in \mathbf{\Sigma}_0 - \mathcal{I}(f)$, which occurs when $\beta_2 \neq 0$. (See Figure 4.1.) The second possibility is $\mathbf{a}_1 = \epsilon_2 \in \mathcal{I}$, which occurs when $\beta_2 = 0$. (See Figure 4.2.) An analysis of the possibilities for $\mathcal{O}_1$ and $\mathcal{O}_2$ will yield the candidates for $|\mathcal{O}_1|$, $|\mathcal{O}_2|$ and $\#\mathcal{L}^{c/o}$, and thus give the possibilities for $\delta(\alpha, \beta)$. We will determine $\delta(\alpha, \beta)$ by finding the possibilities for $\#\mathcal{L}^{c/o}$ and then applying Theorem 3.3.

Theorem 4.2. *If the critical triangle is non-degenerate, $\beta_2 \neq 0$, and $(\alpha, \beta) \notin V_*$, then $\delta_\star \leq \delta(\alpha, \beta) \leq \phi$.*

Proof. Let Y be as in (1.2), and let $f_Y : Y \to Y$ be the induced map. Since $\mathbf{a}_1 = e_2 \neq \epsilon_i$ for $i = 0, 1, 2$, we have

$$f_Y : \Sigma_1 - \mathcal{I} \to \mathbf{a}_1 \to [\beta_2 : \alpha_2]_{E_1} \to [\beta_1\beta_2 : \beta_1\alpha_2 : \alpha_1\beta_2] \in \Sigma_B - \Sigma_0. \tag{4.2}$$

If $f^2\mathbf{a}_1 = f_Y^2\mathbf{a}_1 = \epsilon_0$, then both lines Σ_2 and Σ_B contain ϵ_1, ϵ_0. Since $\mathbf{a}_2 = \Sigma_B \cap \Sigma_1$ and $\epsilon_0 = \Sigma_2 \cap \Sigma_1$, we have $\mathbf{a}_2 = \epsilon_0$. By the second statement of Lemma 1.1, neither $\mathcal{O}_1$ nor $\mathcal{O}_2$ can end at ϵ_2. It follows that we have at most two singular orbits. We have three cases.

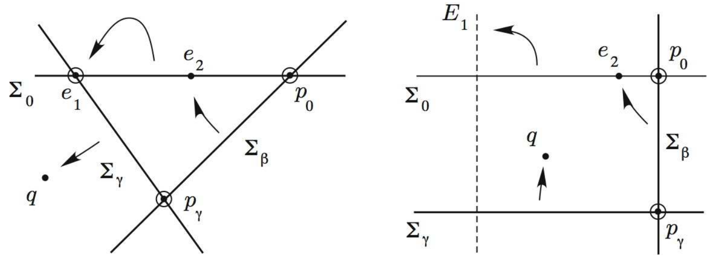


Figure 4.1. Nondegenerate critical triangle: case $\beta_2 \neq 0$.

The first case is where neither $\mathcal{O}_1$ nor $\mathcal{O}_2$ is singular. In this case the orbit list structure is $\#\mathcal{L}^c = \emptyset, \#\mathcal{L}^o = \{\{1\}\}$. By Theoerm 3.3, $\delta(\alpha, \beta)$ is the largest real root of the polynomial

$$\chi(x) = (x-2)x + (x-1) = x^2 - x - 1 \tag{4.3}$$

and is thus equal to ϕ.

In the second case both $\mathcal{O}_0$ and $\mathcal{O}_1$ are singular. In this case the orbit $\mathcal{O}_2$ can not be singular and therefore $f^2\mathbf{a}_1 \neq \epsilon_0$. By the equation (4.2) with above argument, we have $n_1 = |\mathcal{O}_1| \geq 4$ and $\mathcal{O}_1 = \{\mathbf{a}_1, \ldots, \epsilon_0\}$. It follows that $\#\mathcal{L}^o = \emptyset, \#\mathcal{L}^c = \{\{1, n_1\}\}$. The dynamic degree $\delta(\alpha, \beta)$ is the largest root of the polynomial

$$\chi(x) = (x-2)(x^{1+n_1} - 1) + (x-1)(x + x^{n_1} + 2) = x^{n_1}(x^2 - x - 1) + x^2. \tag{4.4}$$

When $n_1 = 4$, the characteristic polynomial is given by $x^6 - x^5 - x^4 + 2 = x^2(x-1)(x^3 - x - 1)$. Thus $\delta = \delta_\star$ in this case. Let us observe that the Comparison Principle [BK, Theorem 5.1] concerns the modulus of the largest zero of the characteristic polynomial of f^*. In §3 we showed that the characteristic polynomials are the same in the elementary and the non-elementary cases. Thus we may apply the Comparison Principle to conclude that $\delta(\alpha, \beta) \geq \delta_\star$ if $n_1 \geq 4$.

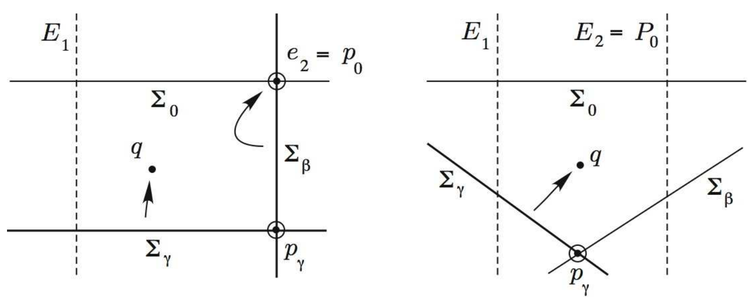


Figure 4.2. Nondegenerate critical triangle: case $\beta_2 = 0$.

The last case is where both $\mathcal{O}_0$ and $\mathcal{O}_2$ are singular. We have $n_2 = |\mathcal{O}_2| \geq 1$ and $\mathcal{O}_2 = \{\mathbf{a}_2, \ldots, \epsilon_0\}$. Therefore the orbit list structure is $\#\mathcal{L}^c = \emptyset, \#\mathcal{L}^o = \{\{n_2, 1\}\}$. By Theorem 3.3, the dynamic degree $\delta(\alpha, \beta)$ is the largest root of the polynomial

$$\chi(x) = (x-2)x^{1+n_2} + (x-1)(x + x^{n_2} + 1) = x^{n_2}(x^2 - x - 1) + x^2 - 1. \tag{4.5}$$

If $n_2 = 1$, we have $\chi(x) = x^3 - x - 1$.

Theorem 4.3. *Assume that the critical triangle is non-degenerate. If $\beta_2 = 0$ and $n_2 = |\mathcal{O}_2| \geq 8$, then $1 < \delta(\alpha, \beta) \leq \delta_\star$. If $\beta_2 = 0$ and $n_2 = |\mathcal{O}_2| \leq 7$, then $\delta(\alpha, \beta) = 1$.*

Proof. If $\beta_2 = 0$, we have $\mathbf{a}_1 = \epsilon_2$ and therefore we have

$$\mathcal{O}_0 = \{\mathbf{a}_0 = \epsilon_1\}, \quad \text{and} \quad \mathcal{O}_1 = \{\mathbf{a}_1 = \epsilon_2\}.$$

If the orbit $\mathcal{O}_2$ is non-singular, we have the orbit list structure $\#\mathcal{L}^o = \{\{1,1\}\}, \#\mathcal{L}^c = \emptyset$. By Theorem 3.3, the degree growth rate $\delta(\alpha,\beta)$ is the largest root of the polynomial

$$\chi(x) = (x-2)x^2 + (x-1)(x+x+1) = x^3 - x - 1. \tag{4.6}$$

If the orbit $\mathcal{O}_2$ is singular, the end point of the orbit has to be the remaining point of indeterminacy, ϵ_0. Thus we have $n_2 = |\mathcal{O}_2| \geq 1$ and $\mathcal{O}_2 = \{\mathbf{a}_2, \dots, \epsilon_0\}$. It follows that the orbit list structure $\#\mathcal{L}^c = \{\{1,1,n_2\}\}, \#\mathcal{L}^o = \emptyset$. Using Theorem 3.3, the dynamic degree is the largest root of the polynomial

$$\begin{aligned}\chi(x) =& (x-2)(x^{2+n_2}-1) + (x-1)(2x^{1+n_2} + x^2 + x^{n_2} + 2x + 3) \\ =& x^{n_2}(x^3 - x - 1) + x^3 + x^2 - 1.\end{aligned} \tag{4.7}$$

It follows from the Comparison Principle ([BK, Theorem 5.1]) that $1 \leq \delta(\alpha,\beta) \leq \delta_\star$. For $n_2 = 7$, we have $\chi(x) = (x^2-1)(x^3-1)(x^5-1)$ and so the $\delta(\alpha,\beta) = 1$. For $n_2 = 8$ we have $\chi(x) = (x-1)(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$ and $\chi'(1) < 0$ and therefore the largest real root is strictly bigger than 1. It follows, then, from the Comparison Principle that $\delta(\alpha,\beta) > 1$ if $n_2 \geq 8$.

Let us note that when the orbit of q lands on p, and we blow up the orbit of q, then we have removed the last exceptional curves for f and f^{-1}. Thus we have:

Proposition 4.4. *If $(\alpha,\beta) \in V_n$, then the induced map $f_X : X \to X$ is biholomorphic.*

Figure 4.3 shows the arrangement of the exceptional varieties in X in the case where the orbit of q does not enter Σ_β.

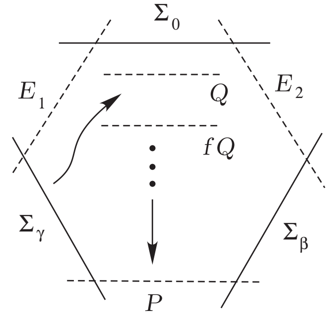


Figure 4.3. Nondegenerate critical triangle; elementary case $(\alpha,\beta) \in V_n$.

Proof of Theorem 1. The statements about degree growth follow from Theorems 4.1–3. It remains only to show that $f_{\alpha,\beta}$ is not birationally conjugate to an automorphism. We consider various cases. First, if $(\alpha,\beta) \in V_*$, then by Theorem 4.1, the degrees of the iterates grow linearly. It follows from [DF, Theorem 0.2] that f is not conjugate to an automorphism. The next case is where the orbit list structure is $\#\mathcal{L}^c = \emptyset$, $\#\mathcal{L}^o = \{\{1\}\}$, i.e., the generic case for non-degenerate triangle, as well as the case of a degenerate triangle. In this case $f_Y : Y \to Y$ is 1-regular, and $f_Y^* = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$ with ordered basis $\{H, E\}$. Then $T = \frac{3+\sqrt{5}}{2}H - E$ represents the cohomology class of the expanded current, and the intersection product is $T \cdot T > 0$. Thus, by [DF, Theorem 0.4] f is not conjugate to an automorphism.

Finally, let us suppose that g is an automorphism. Then the characteristic polynomial of g^* has the form $x^j p(x)$, where $p(x)$ is symmetric, i.e., if r is a root of p, then so is $1/r$. This is an easy consequence of the fact that $(g^{-1})^* = (g^*)^{-1}$ and the result of [DF] that if $\delta(g) > 1$, then $\delta(g)$ is a simple eigenvalue and the unique eigenvalue of modulus greater than one. In particular, the minimal polynomial of $\delta(g)$ (which is a birational invariant) must be symmetric. We have computed the characteristic polynomial for f in all other cases for Theorem 1, and from all of these, we can see that the minimal polynomial of $\delta(g)$ is not symmetric.

§5. Periodic Mappings

Here we determine the precise degree growth rate when $|\mathcal{O}_2| \le 7$. In particular, we show that the degree grows quadratically when $|\mathcal{O}_2| = 7$, and we show that f is periodic when $|\mathcal{O}_2| \le 6$. We do this by showing that f^* is periodic in this case, and then we show that the periodicity of f^* implies the periodicity of f.

Notice that if $|\mathcal{O}_2| = n$, then $f^n(\Sigma_\gamma) = f^{n-1}(q) = p$, and therefore $(\alpha, \beta) \in V_{n-1}$. To show the periodicity of f^*_X it suffices to show that all roots of (4.7) with $n \le 6$ are roots of unity and are simple. For $n \le 6$ we list the characteristic polynomials, together with the smallest polynomials of the form $x^m - 1$ that they divide:

$$
\begin{aligned}
&V_0\ (n=1): && (x-1)(x+1)(x^2+x+1) | (x^6-1)\\
&V_1\ (n=2): && (x-1)(x^4+x^3+x^2+x+1) | (x^5-1)\\
&V_2\ (n=3): && (x-1)(x+1)(x^4+1) | (x^8-1)\\
&V_3\ (n=4): && (x-1)(x^2+x+1)(x^4-x^2+1) | (x^{12}-1)\\
&V_4\ (n=5): && (x-1)(x+1)(x^6-x^3+1) | (x^{18}-1)\\
&V_5\ (n=6): && (x-1)(x^8+x^7-x^5-x^4-x^3+x+1) | (x^{30}-1)
\end{aligned}
$$

Thus we have:

Lemma 5.1. *Assume that the critical triangle is non-degenerate. If $\beta_2 = 0$ and $n = |\mathcal{O}_2| \le 6$, then f^*_X is periodic, with period κ_n, where $\kappa_n = 6, 5, 8, 12, 18, 30$ (respectively).*

When $|\mathcal{O}_2| = 7$, the largest root of equation (4.7) is 1 and has multiplicity 3. The matrix representation from §3 has a 3×3 Jordan block with eigenvalue 1. This means that f^*_X has quadratic growth, and we have:

Lemma 5.2. *Assume that the critical triangle is non-degenerate. If $\beta_2 = 0$ and $|\mathcal{O}_2| = 7$, then f^*_X has quadratic growth.*

Notice that $|\mathcal{O}_2| = 1$ if and only if $q = p_\gamma$, which means that the parameters in V_0 satisfy $\alpha_1\beta_0 - \alpha_0\beta_1 = -\alpha_2\beta_0 = \alpha_1\alpha_2$. With these conditions on α and β, f has a period 6 cycle $\Sigma_\beta \mapsto e_2 \mapsto \Sigma_0 \mapsto e_1 \mapsto \Sigma_\gamma \mapsto p_\gamma \mapsto \Sigma_\beta$, and it is not hard to check that the map f is indeed periodic with period 6.

Theorem 5.3. *Assume that the critical triangle is non-degenerate. if $\beta_2 = 0$ and $|\mathcal{O}_2| \le 6$, then f is periodic with period κ_n.*

To prove Theorem 5.3, we use the following lemma:

Lemma 5.4. *If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is a linear map with five invariant lines such that no more than three of them meet at any point, then f is the identity.*

Proof. Let l_i, $i = 0, 1, 2, 3, 4$, denote the lines fixed by f. Three of these are in general position, so we may assume that $\Sigma_i = \{x_i = 0\}$ for $i = 0, 1, 2$. It follows that f is a linear map represented as a diagonal matrix in the affine coordinates $(x_1/x_0, x_2/x_0)$. One of the lines ℓ_3 or ℓ_4 does not pass through the origin, and f can preserve this line only if it is the identity.

Proof of Theorem 5.3. It suffices to show that f^{κ_n} has at least five invariant lines for $n = 2, \ldots, 6$. Consider the basis elements E_1, E_2, $\mathcal{F}_q$, and $\mathcal{F}_{p_\gamma}$. Since $(f^*_X)^{\kappa_n}$ is the identity, it fixes these basis elements. Thus f^{κ_n} fixes the base points in $\mathbf{P}^2$. Since f^{κ_n} is linear, it leaves invariant every line through two of these base points.

§6. Parameter Regions

There is a natural group action on parameter space. Namely, for $(\lambda, c, \mu) \in \mathbf{C}_* \times \mathbf{C}_* \times \mathbf{C}$ we have actions

$$(\alpha, \beta) \mapsto (\lambda\alpha, \lambda\beta) \tag{6.1}$$

$$(\alpha, \beta) \mapsto (\alpha_0, c\alpha_1, c\alpha_2, c\beta_0, c^2\beta_1, c^2\beta_2) \tag{6.2}$$

$$\begin{aligned}(\alpha, \beta) \mapsto &(\alpha_0 + \mu(\alpha_1 + \alpha_2) - \mu(\beta_0 + \mu(\beta_1 + \beta_2)),\\ &\alpha_1 - \mu\beta_1, \alpha_2 - \mu\beta_2, \beta_0 + \mu(\beta_1 + \beta_2), \beta_1, \beta_2).\end{aligned} \tag{6.3}$$

The first action corresponds to the homogeneity of $f_{\alpha,\beta}$. The other two are given by linear conjugacies of $f_{\alpha,\beta}$. To see them, we write f in affine coordinates, as in (0.2). Action (6.2) is given by conjugating by the scaling map $(x_1, x_2) \mapsto (cx_1, cx_2)$, and (6.3) is given by conjugating by the translation $(x_1, x_2) \mapsto (x_1 + \mu, x_2 + \mu)$.

Now let us comment on maps of the special form

$$f : (x, y) \mapsto (y, \frac{y}{b + x + cy}), \quad b \neq 0. \tag{6.4}$$

In this case we have $\alpha = (0, 0, 1)$, $\beta = (b, 1, c)$ and $\gamma = (0, 0, 1)$. The parameter set V_* defined in the Introduction is the (6.1–3)-orbit of the map (6.4) for the special case $bc = -1$, which is the case of linear degree growth. Let Y be as in (1.2), and let $f_Y : Y \to Y$ be the induced map. Repeating the computation of (1.3), we see that

$$\Sigma_\beta \mapsto E_2 \mapsto [c : 0 : 1]_{e_1} \in E_1 \mapsto [c : 1 : 0] \in \Sigma_\gamma. \tag{6.5}$$

We conclude that the sub-family (6.4) is critically finite the sense that all exceptional curves have finite orbits:

Proposition 6.1. *If f is as in (6.4), then $q = (0, 0)$ is a fixed point, and the exceptional curves are mapped to q. In particular, f_Y is 1-regular.*

Proof. If $c = 0$, then the exceptional locus is Σ_γ; if $c \neq 0$, then both Σ_β and Σ_γ are exceptional. We see from (6.5) that in either case the exceptional curves are mapped to the fixed point.

The variety $V_n \subset \{\beta_2 = 0\}$ corresponds to a dynamical property: an exceptional line is mapped to a point of indeterminacy. Thus V_n is invariant under the actions (6.1–3). For $(\alpha, \beta) \in V_n$, we have $\beta_2 = 0$, and we may apply (6.3) to obtain $\alpha_1 = 0$. Since by (0.4) we must have $\alpha_2 \neq 0$ and $\beta_1 \neq 0$, we apply (6.1) and (6.2) to obtain $\alpha_2 = \beta_1 = 1$. Thus each orbit within V_n is represented by a map which may be written in affine coordinates as

$$(x, y) \mapsto (y, \frac{a + y}{b + x}). \tag{6.6}$$

If f is of the form (6.6), then f^{-1} is conjugate via the involution $\sigma : x \leftrightarrow y$ and a transformation (6.3) to the map

$$(x, y) \mapsto (y, \frac{a - b + y}{-b + x}). \tag{6.7}$$

Such a mapping is conjugate to its inverse if $b = 0$.

Now we suppose that f is given by (6.6). Thus $q = (-a, 0)$ and $p = (-b, -a)$, and V_n is defined by the condition $f^n q = p$. The coefficients of the equations defining V_n are positive integers, and V_n is preserved under complex conjugation. An inspection of the equations defining V_n yields:

V_0: the orbit under (6.1–3) of $(a, b) = (0, 0)$

V_1: the orbit of $(a, b) = (1, 0)$

V_2: the orbits of $(a, b) = ((1 + i)/2, i)$ and its conjugate.

V_3: the orbits of $(a, b) \in \{(2 + i - \sqrt{3})/2, i), (2 + i + \sqrt{3})/2, i)\}$ and their conjugates.

We solve for V_4, V_5 and V_6 by using the resultant polynomials of the defining equations, and we find:

V_4: the orbits of $(a, b) \approx (0.8711 + 0.7309i, 1.4619i)$, $(0.6974 + 0.2538i, 0.5077i)$, $(-0.06857 + 0.3889i, 0.7778i)$, and their conjugates. The exact values are roots of $1 - 3\,a + 9a^2 - 24a^3 + 36a^4 - 27a^5 + 9a^6$ and $1 + 6b^2 + 9b^4 + 3b^6$.

V_5: the orbits of $(a, b) \approx (3.7007 + 1.2024, 2.4048i)$, $(1.0353 + 0.3364i, 0.6728i)$, $(0.4465 + 0.6146i, 1.2293i)$, $(-0.1826 + 0.2513i, 0.5027i)$, or their conjugates. The exact values are roots of $1 + 3a^2 - 20a^3 + 49a^4 - 60a^5 + 37a^6 - 10\,a^7 + a^8$ and $1 + 7b^2 + 14b^4 + 8b^6 + b^8$.

V_6: The defining equations for V_6 are divisible by b^2, so all points of the form $(a, 0)$, $a \neq 0, 1$, belong to V_6. By (6.7), these parameters correspond to maps which are conjugate to their inverses. In addition, V_6 contains the orbits of

$$a = (3 \pm \sqrt{5} + 2b)/4, \quad b = i\sqrt{(5 \pm \sqrt{5})/2}$$

and their conjugates.

By Theorem 2, mappings in V_6 have quadratic degree growth, and by [G] such mappings have invariant fibrations by elliptic curves. Let us show how our approach yields these invariant fibrations.

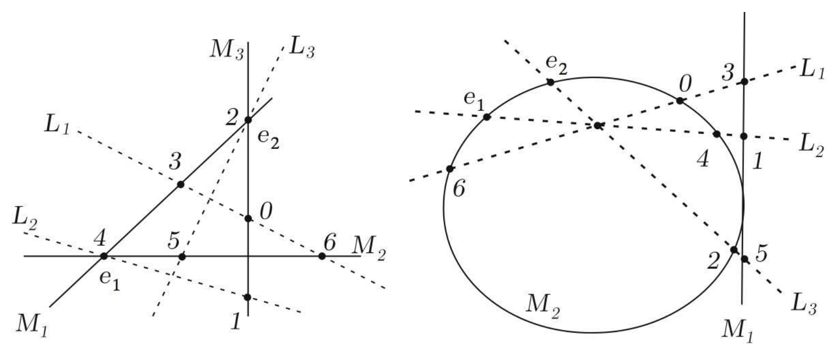


Figure 6.1. Points $f^j q =$ 'j', $0 \le j \le 6$, for V_6. Case $b = 0$ on left; $b \neq 0$ on right.

Let us first consider parameters $(a, 0)$. In this case, the fibration was obtained classically in [L] and [KoL]. In the space Y of (1.2), the f-orbit $\{q_j = f^j q : j = 0, 1, \ldots, 6\}$ is:

$$q_0 = (-a, 0)_{\mathbf{C}^2} = [1 : -a : 0],\ q_1 = (0, -1)_{\mathbf{C}^2} = [1 : 0 : -1],\ q_2 = [0 : 0 : 1] = e_2,$$
$$q_3 = [0 : 1 : -1],\ q_4 = [1 : 0 : -1]_{e_1},\ q_5 = (-1, 0),\ q_6 = (0, -a) = p,$$

as is shown in Figure 6.1. Here we use 'j' to denote 'q_j'. The construction of X is shown in Figure 6.2, where '$f^j Q$' denotes the blowup fiber over q_j. In contrast, the case corresponding to $(a, b) \in V_6$, $b \neq 0$ corresponds to Figure 4.3. Consulting Figure 6.2, we see that the cohomology class $3H_X - E_1 - E_2 - Q_2 - Q_4 - \sum Q_j$ is fixed under f^*. We will find polynomials which correspond as closely as possible to this class. These will be cubics which vanish on e_i and q_j. Looking for lines that contain as many of the q_j as possible, we see $L_1 = \{x + y + a = 0\}$ contains 0,3,6. Mapping forward by f, we have

$$L_1 \mapsto L_2 = \{y + 1 = 0\} \mapsto L_3 = \{x + 1 = 0\} \mapsto L_1.$$

In addition, the points q_j, $j = 2, 3, 4$ are contained in the line at infinity $M_1 = \Sigma_0$. This maps forward as:

$$M_1 \mapsto e_1 \mapsto M_2 = \{y = 0\} \mapsto M_3 = \{x = 0\} \mapsto e_2 \mapsto M_1.$$

The cubic $c_1 = (x+y+at)(x+t)(y+t)$ defines $L_1+L_2+L_3$ in $\mathbf{P}^2$, and $c_2 = xyt$ defines $M_1+M_2+M_3$. Setting $t = 1$ and taking the quotient, we find the classical invariant $h(x, y) = c_1/c_2$.

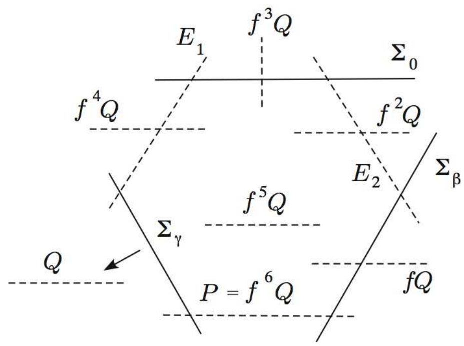


Figure 6.2. Space X for V_6, $b = 0$.

Now we consider the other four parameters (a, b) in V_6. Inspecting the defining equations of V_6, we find that a and b satisfy $-2a + a^2 + b - ab = 0$ and $-b^2 - 1 + b - 2a = 0$. Using these relations, we see that the f-orbit of q is:

$$q_0 = (-a, 0),\ q_1 = (0, 1 - a),\ q_2 = (1 - a, 1/b),\ q_3 = (1/b, a(1 + ab)/(ab - b^2)),$$
$$q_4 = (a(1 + ab)/(ab - b^2), 1 - a),\ q_5 = (1 - a, -b),\ q_6 = (-b, -a).$$

Looking again at the points q_j, $j = 0, 3, 6$, we see that they are contained in a line $L_1 = \{x + (1 - \frac{b}{a})y + a = 0\}$. Mapping L_1 forward under f, we find:

$$L_1 \mapsto L_2 = \{y + a - 1 = 0\} \mapsto L_3 = \{x + a - 1 = 0\} \mapsto L_1.$$

We multiply these linear functions together to obtain a cubic c_1 which defines $\sum L_i$. We see, too, that the points q_j, $j = 1, 3, 5$ are contained in the line $M_1 = \{(a-b-1)x + (a-1)y + (a-1)^2 = 0\}$. Mapping forward, we find:

$$M_1 \mapsto M_2 = \{(a - 1)xy + (b^2 + 1)y + (a - b - 1)x + (a - b) = 0\} \mapsto M_1.$$

Multiplying the defining functions, we obtain a cubic c_2 which defines $M_1 + M_2$. Now we define $k(x, y) = c_1/c_2$. And inspection shows that $k \circ f = \omega k$, where ω is a 5th root of unity. Thus f is a period 5 mapping of the set of cubics $\{k = \text{const}\}$ to itself.

Appendix: Explanation of the Computer Graphics

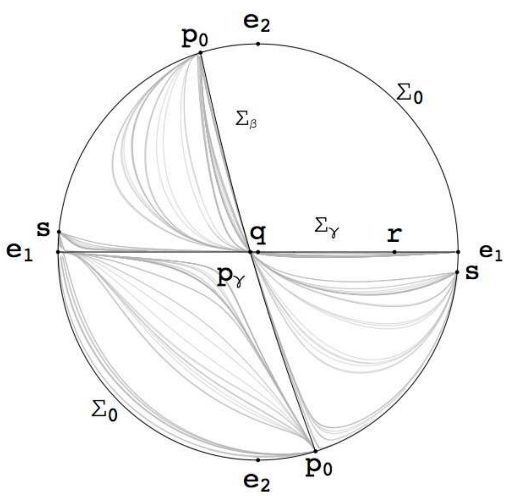
It is useful to have visual representations for rational mappings. A number of interesting computer graphic representations of the behavior of rational mappings of the real plane have been given in various works by Bischi, Gardini and Mira; we cite [BGM] as an example. The pictures here have a somewhat different origin and are made following a scheme used earlier by one of the authors and Jeff Diller (see [BD1,3]). They are motivated by the theory of dynamics of complex surface maps. Let f be a birational map of a Kähler surface. If $\delta(f) > 1$, then there are positive, closed, (1,1)-currents $T^{\pm}$ such that $f^*T^+ = \delta(f)T^+$ and $f^*T^- = \delta(f)^{-1}T^-$ (see Diller-Favre [DF]). These currents have the additional property that for any complex curve Γ there is a number $c > 0$ such that

$$cT^+ = \lim_{n\to\infty} \frac{1}{\delta^n} f^{n*}[\Gamma], \tag{A.1}$$

and similarly for T^-. By work of Dujardin [D1] these currents have the structure of a generalized lamination. We let $\mathcal{L}^{s/u}$ denote the generalized laminations corresponding to $T^{\pm}$. It was shown in [BD2] that the wedge product $T^+ \wedge T^-$ defines an invariant measure in many cases, and Dujardin [D2] showed that this invariant measure may be found by taking the "geometric intersection" of the measured laminations $\mathcal{L}^s$ and $\mathcal{L}^u$.

When one of our mappings f has real coefficients, it defines a birational map of the real plane, and we can hope that there might be real analogues for the results of the theory of complex surfaces. This was proved to be the case for certain maps in [BD1,3] but is not known to hold for the maps studied in the present paper.

Figure 0.1 was drawn as follows. We work in the affine coordinate chart (x, y) on $\mathbf{R}^2$ given by $x_0 = 1$, $x = x_1/x_0 = x_1$, $y = x_2/x_0 = x_2$. We start with a long segment $L \subset \mathbf{R}^2$ and map it forward several times. The resulting curve is colored black and "represents" $\mathcal{L}^u$. After the first few iterates, the computer picture seems to "stabilize," and further iteration serves to "fill out" the lamination. The appearance of the computer picture obtained in this manner is independent of the choice of initial line L. To represent $\mathcal{L}^s$, we repeat this procedure for f^{-1} and color the resulting picture gray. In Figure 0.1 we present $\mathcal{L}^s$ in gray in the left hand frame. Then we present $\mathcal{L}^s$ and $\mathcal{L}^u$ together in the right hand frame in order to show the set where they intersect.



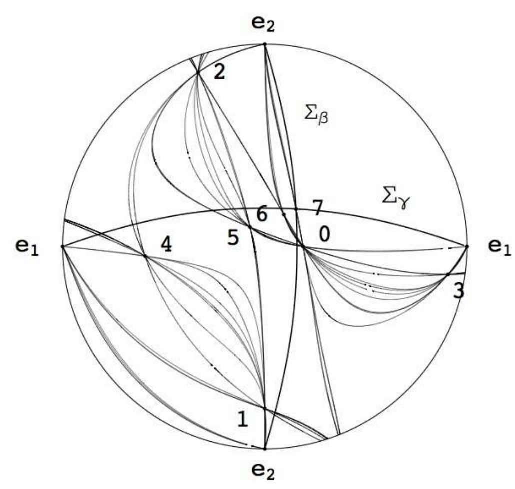


Figure A.1. Explanation of Figure 0.1 (left); a mapping from V_7 (right).

We also want the graphic to have the appearance of a subset of $\mathbf{P}^2$, so we rescale the distance to the origin. The resulting "disk" is a compactification of $\mathbf{R}^2$. In fact, this is real projective space, since antipodal points of the circle are identified. The circle forming the boundary of this disk is the line at infinity Σ_0.

Figure 0.1 is obtained using the map of the form (6.4):

$$(x, y) \mapsto (y, \frac{y}{.1 + x + .3y}).$$

By Proposition 6.1, f is critically finite, so $\delta(f) = \phi$ by Theorem 4.2. On the left half of Figure A.1, we have re-drawn $\mathcal{L}^s$, together with the points of indeterminacy of f and f^{-1}. Pictured, for instance, are e_1, e_2, $p_0 = [0 : -.3 : 1]$, $p_\gamma = (-.1, 0)$, and $q = (0, 0)$. The exceptional curves are lines connecting certain pairs of these points and may be found easily using Figure 1.1 as a guide. As we expect, $\mathcal{L}^s$ is "bunched" at the points of indeterminacy of f, i.e., p_0, e_1, and p_γ. Let us track the backward orbits of these points. First, $p_0 = f^{-1}p_0$ is fixed under f^{-1}, and $f^{-1}p_\gamma = e_1$. Now let Y and f_Y be as in (1.2). Repeating the calculations at equation (1.3), we see that f_Y^{-1} takes p_γ to the fiber point $[1 : 0 : -.1]_{E_1}$ over e_1. Then this fiber point is mapped under f^{-1} to the point $s = [0 : 1.03 : -.1] \in \Sigma_0$. The next preimage is $f^{-1}s = p_0$, so f^{-1} is critically finite in the sense that the exceptional curves all have finite orbits. This explains why $\mathcal{L}^s$ is "bunched" at only four points.

To explain the points where $\mathcal{L}^u$ is "bunched," we have plotted the point $r := f^3\Sigma_\beta = (10/3, 0)$ from (6.5). If we superimpose the picture of $\mathcal{L}^u$ on the left panel of Figure A.1, we find that $\mathcal{L}^u$ is "bunched" exactly on the set e_1, e_2, q, and r. The "eye" which appears in the first quadrant is due to an attracting fixed point.

The right hand side of Figure A.1 is obtained using the map

$$(x, y) \mapsto (y, \frac{-.499497 + y}{-.415761 + x}),$$

which corresponds to a real parameter $(a, b) \in V_7$. By "j", $j = 0, \ldots, 7$, we denote the point $f^j q$. Thus "7" is the point of indeterminacy $p = f^7 q$. We let $\pi : X \to \mathbf{P}^2$ be the manifold obtained by blowing up e_1, e_2, and "j" for $j = 0, \ldots, 7$. The lamina of $\mathcal{L}^u$ are then separated in X, and the apparent intersections may be viewed as artifacts of the projection π.

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