# Dynamics of Rational Surface Automorphisms: Linear Fractional Recurrences

Eric Bedford and Kyounghee Kim

**§0. Introduction.** Here we discuss automorphisms (biholomorphic maps) of compact, projective surfaces with positive entropy. Cantat [C1] has shown that the only possibilities occur for tori and $K3$ (and certain of their quotients), and rational surfaces. $K3$ surfaces have been studied by Cantat [C2] and McMullen [M1]. Here we consider the family of birational maps of the plane which are defined by

$$f_{a,b} : (x,y) \mapsto \left(y, \frac{y+a}{x+b}\right), \tag{0.1}$$

and which provide an interesting source of automorphisms of rational surfaces. The maps $f_{a,b}$ form part of the family of so-called linear fractional recurrences, which were studied from the point of view of degree growth and periodicity in [BK]. We let $\mathcal{V} = \{(a,b) \in \mathbf{C}^2\}$ be the space of parameters for this family, and we define

$$\begin{aligned} &q = (-a, 0),\ p = (-b, -a), \\ &\mathcal{V}_n = \{(a,b) \in \mathcal{V} : f^j_{a,b} q \neq p \text{ for } 0 \leq j < n, \text{ and } f^n_{a,b} q = p\}. \end{aligned} \tag{0.2}$$

In [BK] we showed that $f_{a,b}$ is birationally conjugate to an automorphism of a compact, complex surface $\mathcal{X}_{a,b}$ if and only if $(a,b) \in \mathcal{V}_n$ for some $n \geq 0$. The surface $\mathcal{X}_{a,b}$ is obtained by blowing up the projective plane $\mathbf{P}^2$ at the $n+3$ points $e_1 = [0:1:0]$, $e_2 = [0:0:1]$, and $f^j q$, $0 \leq j \leq n$. The dimension of $Pic(\mathcal{X}_{a,b})$ is thus $n+4$, and $f^*_{a,b}$ is the same for all $(a,b) \in \mathcal{V}_n$; the characteristic polynomial is

$$\chi_n(x) := -1 + x^2 + x^3 - x^{1+n} - x^{2+n} + x^{4+n}. \tag{0.3}$$

When $n \geq 7$, $\chi_n$ has a unique root $\lambda_n > 1$ with modulus greater than one, and the entropy of $f_{a,b}$ is $\log \lambda_n > 0$. More recently, McMullen [M2] considers automorphisms of blowups of $\mathbf{P}^2$ in terms of their action on the Picard group and connects the numbers $\lambda_n$ with the growth rates of Coxeter elements of Coxeter groups; [M2] then gives a synthesis of surface automorphisms starting from $\lambda_n$ (or one of its Galois conjugates) and an invariant cubic. This gives an alternative construction of certain of the maps of $\mathcal{V}_n$ and in particular shows that $\mathcal{V}_n$ is nonempty for all values of $n$ (a question left open in [BK]).

The purpose of this paper is to further discuss the maps $f_{a,b}$ for $(a,b) \in \mathcal{V}_n$, $n \geq 7$. In §1,2 we show that the maps with invariant cubics are divided into three families, $\Gamma_j \subset \mathcal{V}$, $j = 1,2,3$, which correspond to maps constructed in [M2]. In §2 we describe the intersection $\Gamma_j \cap \mathcal{V}_n$. In §3 we show the existence of automorphisms $f_{a,b}$ of $\mathcal{X}_{a,b}$ without invariant curves. In particular, these are examples of rational surfaces with automorphisms of positive entropy, for which the pluri-anticanonical bundle has no nontrivial sections. This gives a negative answer to a conjecture/question of Gizatullin, Harbourne, and McMullen.

We next describe the dynamics of the maps $f_{a,b}$ for $(a,b) \in \Gamma \cap \mathcal{V}_n$ for $n \geq 7$. We discuss rotation (Siegel) domains in §4. In §5 we look at the real mappings in the families $\Gamma_j$, i.e., $(a,b) \in \Gamma_j \cap \mathbf{R}^2$. The set $\mathcal{X}_R$ of real points inside $\mathcal{X}_{a,b}$ is then invariant under $f_{a,b}$. Let $f_R$ denote the restriction of $f_{a,b}$ to $\mathcal{X}_R$. We show in §5 that $f_R$ has maximal entropy in the sense that its entropy is equal to the entropy of the complex map $f_{a,b}$. The condition that $f_R$ has maximal entropy has been useful in several cases to reach a deeper understanding of the map (see [BD1,2] and [BS2]).

Among the maps with invariant curves, there is a dichotomy:

**Main Theorem.** *Let $f_{a,b}$ be a mapping of the form (0.1) which is equivalent to an automorphism with positive entropy. If there is an $f_{a,b}$-invariant curve, then one of the following occurs:*

*(i) $f$ has a rotation (Siegel) domain centered at a fixed point.*

*(ii) $a, b \in \mathbf{R}$, and $f_R$ has the same entropy as $f$. Further the (unique) invariant measure of maximal entropy is supported on a subset of $\mathcal{X}_R$ of zero area.*

The proof is given in §6. The mappings of $\mathcal{V}_6$ all have invariant pencils of cubics and entropy zero. The relation between the family $\mathcal{V}_6$ and the curves $\Gamma$ are discussed in Appendix A. Then, in Appendix B, we discuss the relation between $\mathcal{V}_n \cap \Gamma$ and $\mathcal{V}_n - \Gamma$ for the case of positive entropy. Finally, in Appendix C we give an auxiliary calculation of characteristic polynomials, which is used in §5.

Acknowledgement. We wish to thank Serge Cantat and Jeff Diller for explaining some of this material to us and giving helpful suggestions on this paper.

## §1. Invariant Curves.

We define $\alpha = \gamma = (a, 0, 1)$ and $\beta = (b, 1, 0)$, so in homogeneous coordinates the map $f$ is written

$$f_{a,b} : \ [x_0 : x_1 : x_2] \mapsto [x_0 \beta \cdot x : x_2 \beta \cdot x : x_0 \alpha \cdot x].$$

The exceptional curves for the map $f$ are given by the lines $\Sigma_0 = \{x_0 = 0\}$, $\Sigma_\beta = \{\beta \cdot x = 0\}$, and $\Sigma_\gamma = \{\gamma \cdot x = 0\}$. The indeterminacy locus $\mathcal{I}(f) = \{e_2, e_1, p\}$ consists of the vertices of the triangle $\Sigma_0 \Sigma_\gamma \Sigma_\beta$. Let $\pi : \mathcal{Y} \to \mathbf{P}^2$ be the complex manifold obtained by blowing up $e_1$ and $e_2$, and let the exceptional fibers be denoted $E_1$ and $E_2$. By $\Sigma_0$, $\Sigma_\beta$ and $\Sigma_\gamma$ we denote the strict transforms in $\mathcal{Y}$. Let $f_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y}$ be the induced birational map. Then the exceptional locus is $\Sigma_\gamma$, and the indeterminacy loci are $\mathcal{I}(f_\mathcal{Y}) = \{p\}$ and $\mathcal{I}(f_\mathcal{Y}^{-1}) = \{q\}$. In particular, $f_\mathcal{Y} : \Sigma_\beta \to E_2 \to \Sigma_0 \to E_1 \to \Sigma_B = \{x_2 = 0\}$. By curve, we mean an algebraic set of pure dimension 1, which may or may not be irreducible or connected. We say that an algebraic curve $S$ is invariant if the closure of $f(S - \mathcal{I})$ is equal to $S$. We define the cubic polynomial $j_f := x_0(\beta \cdot x)(\gamma \cdot x)$, so $\{j_f = 0\}$ is the exceptional locus for $f$. For a homogeneous polynomial $h$ we consider the condition that there exists $t \in \mathbf{C}^*$ such that

$$h \circ f = t \cdot j_f \cdot h. \tag{1.1}$$

**Proposition 1.1.** *Suppose that $(a,b) \notin \bigcup \mathcal{V}_n$, and $S$ is an $f_{a,b}$-invariant curve. Then $S$ is a cubic containing $e_1, e_2$, as well as $f^j q, f^{-j} p$ for all $j \geq 0$. Further, (1.1) holds for $S$.*

*Proof.* Let us pass to $f_\mathcal{Y}$, and let $S$ denote its strict transform inside $\mathcal{Y}$. Since $(a,b) \notin \bigcup \mathcal{V}_n$, the backward orbit $\{f^{-n} p : n \geq 1\}$ is an infinite set which is disjoint from the indeterminacy locus of $\mathcal{I}(f_{\mathcal{Y}}^{-1})$. It follows that $S$ cannot be singular at $p$ (cf. Lemma 2.3 of [DJS]). Let $\mu$ denote the degree of $S$, and let $\mu_1 = S\cdot E_1$, $\mu_0 = S\cdot\Sigma_0$, and $\mu_2 = S\cdot E_2$. It follows that $\mu = \mu_2+\mu_0+\mu_1$. Further, since $f_{\mathcal{Y}} : E_2 \to \Sigma_0 \to E_1$, we must have $\mu_2 = \mu_0 = \mu_1$. Thus $\mu$ must be divisible by 3, and $\mu/3 = \mu_2 = \mu_0 = \mu_1$. Now, since $S$ is nonsingular at $p = \Sigma_\beta \cap \Sigma_\gamma$, it must be transversal to either $\Sigma_\gamma$ or $\Sigma_\beta$.

Let us suppose first that $S$ is transverse to $\Sigma_\gamma$ at $p$. Then we have $\mu = \mu_2 + S\cdot\Sigma_\gamma$. Thus $S$ intersects $\Sigma_\gamma - \{p\}$ with multiplicity $\mu-(\mu/3)-1$. If $\mu>3$, then this number is at least 2. Now $\Sigma_\gamma$ is exceptional, $f_{\mathcal{Y}}$ is regular on $\Sigma_\gamma - \{p\}$, and $f_{\mathcal{Y}}(\Sigma_\gamma - \{p\}) = q$. We conclude that $S$ is singular at $q$. This is not possible by Lemma 2.3 of [DJS] since $q$ is indeterminate for $f^{-1}$. This is may also be seen because since $(a,b)\notin\bigcup\mathcal{V}_n$, it follows that $f^nq$ is an infinite orbit disjoint from the indeterminacy point $p$, which is a contradiction since $S$ can have only finitely many singular points.

Finally, suppose that $S$ is transversal to $\Sigma_\beta$. We have $\mu = S\cdot\Sigma_\beta + \mu_2$, and by transversality, this means that $S$ intersects $\Sigma_\beta - \{p\}$ with multiplicity $\frac{2\mu}{3}-1$. On the other hand, $f_{\mathcal{Y}}$ is regular on $\Sigma_\beta - \{p\}$, and $\Sigma_\beta - \{p\} \to E_2$. Thus the multiplicity of intersection of $S$ with $\Sigma_\beta - \{p\}$ must equal the multiplicity of intersection with $E_2$, but this is not consistent with the formulas unless $\mu = 3$. □

The following was motivated by [DJS]:

**Theorem 1.2.** *Suppose that $(a,b)\in\mathcal{V}_n$ for some $n\ge 11$. If $S$ is an invariant algebraic curve, then the degree of $S$ is 3, and (1.1) holds.*

*Proof.* Let $\mathcal{X}$ be the manifold $\pi:\mathcal{X}\to\mathbf{P}^2$ obtained by blowing up $e_1, e_2, q, fq, \dots, f^nq = p$, and denote the blowup fibers by $E_1, E_2, Q, fQ, \dots, f^nQ = P$. Suppose that $S$ is an invariant curve of degree $m$. By $S$, $\Sigma_0$, etc., we denote the strict transforms of these curves inside $\mathcal{X}$. Let $f_{\mathcal{X}}$ be the induced automorphism of $\mathcal{X}$, so $S$ is again invariant for $f_{\mathcal{X}}$, which we write again as $f$. Let us write the various intersection products with $S$ as: $\mu_1 = S\cdot E_1$, $\mu_0 = S\cdot\Sigma_0$, $\mu_2 = S\cdot E_2$, $\mu_P = S\cdot P$, $\mu_\gamma = S\cdot\Sigma_\gamma$, $\mu_Q = S\cdot Q$. Since $e_1, e_2\in\Sigma_0$, we have

$$\mu_1+\mu_0+\mu_2 = m.$$

Now we also have

$$f:\quad \Sigma_\beta \to E_2 \to \Sigma_0 \to E_1$$

so $\mu_\beta = \mu_2 = \mu_0 = \mu_1 = \mu$ for some positive integer $\mu$, and $m = 3\mu$. Similarly, $p, e_2\in\Sigma_\beta$, so we conclude that $\mu_P + \mu_\beta + \mu_2 = m$, and thus $\mu_P = \mu$. Following the backward orbit of $P$, we deduce that $S\cdot f^jQ = \mu$ for all $0\le j\le n$.

Now recall that if $L\in H^{1,1}(\mathbf{P}^2,\mathbf{Z})$ is the class of a line, then the canonical class of $\mathbf{P}^2$ is $-3L$. Thus the canonical class $K_{\mathcal{X}}$ of $\mathcal{X}$ is $-3L+\sum E$, where sum is taken over all blowup fibers $E$. In particular, the class of $S$ in $H^{1,1}(\mathcal{X})$ is $-\mu K_{\mathcal{X}}$. Since we obtained $\mathcal{X}$ by performing $n+3$ blowups on $\mathbf{P}^2$, the genus formula, applied to the strict transform of $S$ inside $\mathcal{X}$, gives:

$$g(S) = \frac{S\cdot(S+K_{\mathcal{X}})}{2}+1 = \frac{\mu(\mu-1)}{2}K_{\mathcal{X}}^2+1 = \frac{\mu(\mu-1)}{2}(9-(3+n))+1.$$

Now let $\nu$ denote the number of connected components of $S$ (the strict transform inside $\mathcal{X}$). We must have $g(S)\ge 1-\nu$. Further, the degree $3\mu$ of $S$ must be at least as large as

$\nu$, which means that $\mu(\mu-1)(n-6) \le 2\nu \le 6\mu$ and therefore $\mu \le 6/(n-6)+1$. We have two possibilities: (i) If $n \ge 13$, then $\mu = 1$, and $S$ must have degree 3; (ii) if $n = 11$ or 12, either $\mu = 1$, (i.e., the degree of $S$ is 3), or $\mu = 2$. Let us suppose $n = 11$ or 12 and $\mu = 2$. From the genus formula we find that $5 \le \nu \le 6$. We treat these two cases separately.

Case 1. $S$ cannot have 6 connected components. Suppose, to the contrary, $\nu = 6$. First we claim that $S$ must be minimal, that is, we cannot have a nontrivial decomposition$S = S_1 \cup S_2$, where $S_1$ and $S_2$ are invariant. By the argument above, $S_1$ and $S_2$ must be cubics, and thus they must both contain all $n+3 \ge 14$ points of blowup. But then they must have a common component, so $S$ must be minimal.

Since the degree of $S$ is 6, it follows that $S$ is the union of 6 lines which map $L_1 \to L_2 \to \cdots \to L_6 \to L_1$. Further, each $L_i$ must contain exactly one point of indeterminacy, since it maps forward to a line and not a quadric. Since the class of $S$ in $H^{1,1}(\mathcal{X})$ is $-2K_{\mathcal{X}}$, we see that $e_1, e_2, p, q \in S$ with multiplicity 2. Without loss of generality, we may assume that $e_1 \in L_1$, which means $\Sigma_\beta \cap L_1 \ne \emptyset$, and therefore $e_2 \in L_2$. Similarly $q \in L_3$. Since the backward image of $L_3$ is a line, $e_1, e_2 \notin L_3$, and thus $p \in L_3$, which gives $\Sigma \cap L_3 \ne \emptyset$. Continuing this procedure, we end up with $L_6 \ni p, q$. It follows that $L_1 = L_4$, $L_2 = L_5$, and $L_3 = L_6$, so $S$ has only 3 components.

Case 2. $S$ cannot have 5 connected components. Suppose, to the contrary, that $\nu = 5$. It follows that $S$ is a union of 4 lines and one quadric. Without loss of generality we may assume that $L_1 \to Q \to L_2 \to L_3$. Since $L_1$ maps to a quadric, it cannot contain a point of indeterminacy, which means that $L_1 \cap \Sigma_0 \ne \emptyset$, $L_1 \cap \Sigma_\beta \ne \emptyset$, and $L_1 \cap \Sigma_\gamma \ne \emptyset$. It follows that $e_1, e_2, q \in Q$. On the other hand, since $L_2$ maps to a quadric by $f^{-1}$, we have $e_1, e_2, p \in Q$, and $e_1, e_2, q \notin L_2$. Thus we have that $Q \cap \Sigma_0 = \{e_1, e_2\}$, $Q \cap \Sigma_\beta = \{e_2, p\}$, and $Q \cap \Sigma_\gamma = \{e_1, p\}$. It follows that $q \notin L_2$, which means that $L_2$ does not contain any point of indeterminacy, and therefore $L_2$ maps to a quadric.

Thus we conclude that $S$ has degree 3, so we may write $S = \{h = 0\}$ for some cubic $h$. Since the class of $S$ in $H^{1,1}(\mathcal{X})$ is $-3K_{\mathcal{X}}$, we see that $e_1, e_2, q \in S$. Since these are the images of the exceptional lines, the polynomial $h \circ f$ must vanish on $\Sigma_0 \cup \Sigma_\gamma \cup \Sigma_\beta$. Thus $j_f$ divides $h \circ f$, and since $h \circ f$ has degree 6, we must have (1.1). □

**Remarks.** (a) From the proof of Theorem 1.2, we see that if $S$ is an invariant curve, $n \ge 11$, then $S$ contains $e_1$, $e_2$, and $f^j q$, $0 \le j \le n$. (b) The only positive entropy parameters which are not covered in Theorem 1.2 are the cases $n = 7, 8, 9, 10$. By Proposition B.1, we have $\mathcal{V}_n \subset \Gamma$ for $7 \le n \le 10$.

**Corollary 1.3.** *If $S$ is $f$-periodic with period $k$, and if $n \ge 11$, then $S \cup \cdots \cup f^{k-1} S$ is invariant and thus a cubic.*

§**2. Invariant Cubics.** In this section, we identify the parameters $(a, b) \in \mathcal{V}$ for which $f_{a,b}$ has an invariant curve, and we look at the behavior of $f_{a,b}$ on this curve. We define the functions:

$$\begin{aligned} \varphi_1(t) &= \left( \frac{t - t^3 - t^4}{1 + 2t + t^2}, \frac{1 - t^5}{t^2 + t^3} \right), \\ \varphi_2(t) &= \left( \frac{t + t^2 + t^3}{1 + 2t + t^2}, \frac{-1 + t^3}{t + t^2} \right), \quad \varphi_3(t) = \left( 1 + t, t - t^{-1} \right), \end{aligned} \tag{2.1}$$

The proofs of the results in this section involve some calculations that are possible but tedious to do by hand; but they are not hard with the help of Mathematica or Maple.

**Theorem 2.1.** *Let $t \neq 0, \pm 1$ with $t^3, t^5 \neq 1$ be given. Then there is a homogeneous cubic polynomial $P$ satisfying (1.1) if and only if $(a,b) = \varphi_j(t)$ for some $1 \leq j \leq 3$. If this occurs, then (up to a constant multiple) $P$ is given by (2.2) below.*

*Proof.* From the proof of Theorem 1.2, we know that $P$ must vanish at $e_1, e_2, q, p$. Using the conditions $P(e_1) = P(e_2) = P(q) = 0$, we may set

$$\begin{aligned} P[x_0 : x_1 : x_2] =& (-a^2C_1 + aC_2)x_0^3 + C_2x_1x_0^2 + C_3x_2x_0^2 + C_1x_0x_1^2 + \\ & + C_4x_2x_1^2 + C_5x_0x_2^2 + C_6x_1x_2^2 + C_7x_0x_1x_2 \end{aligned}$$

for some $C_1, \ldots, C_7 \in \mathbf{C}$. Since $e_1, e_2, q \in \{P = 0\}$, we have $P \circ f = j_f \cdot \tilde{P}$ for some cubic $\tilde{P}$. A computation shows that

$$\begin{aligned} \tilde{P} =& (-ab^2C_1 + b^2C_2 + bC_3 + aC_5)x_0^3 + (-2abC_1 + 2bC_2 + C_3)x_1x_0^2 + \\ & (bC_1 + C_5 + aC_6 + bC_7)x_2x_0^2 + (-aC_1 + C_2)x_0x_1^2 + C_1x_2x_1^2 + (bC_4 + C_6)x_0x_2^2 + \\ & + C_4x_1x_2^2 + (2bC_1 + C_7)x_0x_1x_2. \end{aligned}$$

Now setting $\tilde{P} = tP$ and comparing coefficients, we get a system of 8 linear equations in $C_1, \ldots, C_7$ of the form

$$M \cdot [x_0^3, x_1x_0^2, x_2x_0^2, x_0x_1^2, x_0x_2^2, x_0x_1x_2, x_1x_2^2, x_1^2x_2]^t = 0.$$

We check that there exist cubic polynomials satisfying (1.1) if and only if the two principal minors of $M$ vanish simultaneously, which means that

$$a + abt + abt^4 - b^2t^4 - at^5 + bt^5 = 0$$

$$-1 + (1-a-b)t + (a+b)t^2 + b^2t^3 + b^2t^4 + (a-2b)t^5 + (1-a+2b)t^6 - t^7 = 0$$

Solving these two equations for $a$ and $b$, we obtain $\varphi_j$, $j = 1,2,3$ as the only solutions, and then solving $M = 0$ we find that $P$ must have the form:

$$\begin{aligned} P_{t,a,b}(x) =& ax_0^3(-1+t)t^4 + x_1x_2(-1+t)t(x_2 + x_1t) \\ & + x_0[2bx_1x_2t^3 + x_1^2(-1+t)t^3 + x_2^2(-1+t)(1+bt)] \\ & + x_0^2(-1+t)t^3[a(x_1 + x_2t) + t(x_1 + (-2b+t)x_2)] \end{aligned} \tag{2.2}$$

which completes the proof. □

**Remark.** In §A, we discuss maps in $\mathcal{V}_n$, $0 \leq n \leq 6$. These are maps with invariant cubics, but $(a,b) \neq \varphi_j(t)$. The invariant cubics transform according to (1.1), but the transition factor $t$ is a root of unity excluded in the hypotheses of Theorem 2.1.

If $h$ satisfies (1.1), then we may define a meromorphic 2-form $\eta_P$ on $\mathbf{P}^2$ by setting $\eta_h := \frac{dx \wedge dy}{h(1,x,y)}$ on the affine coordinate chart $[1 : x : y]$. Then $\eta_h$ satisfies $t\, f^*\eta_h = \eta_h$. It follows that if the points $\{p_1, \ldots, p_k\}$ form a $k$-cycle which is disjoint from $\{h = 0\}$, then the Jacobian determinant of $f$ around this cycle will be $t^{-k}$.

Let $\Gamma_j = \{(a, b) = \varphi_j(t) : t \in \mathbf{C}\} \subset \mathcal{V}$ denote the curve corresponding to $\varphi_j$, and set $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Consistent with [DJS], we find that the cases $\Gamma_j$ yield cubics with cusps, lines tangent to quadrics, and three lines passing through a point.

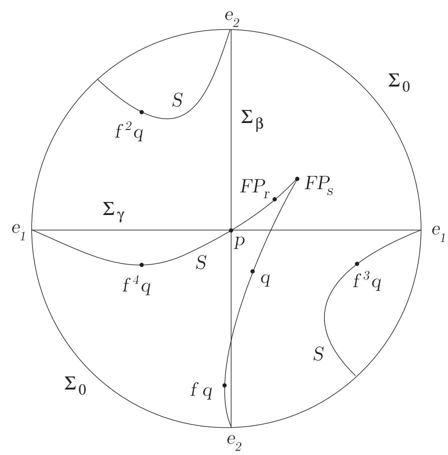


Figure 2.1. Orbit of $q$ for family $\Gamma_1$; $1 < t < \delta_\star$.

$\Gamma_1$*: Irreducible cubic with a cusp.* To discuss the family $\Gamma_1$, let $(a, b) = \varphi_1(t)$ for some $t \in \mathbf{C}$. Then the fixed points of $f_{a,b}$ are $FP_s = (x_s, y_s)$, $x_s = y_s = t^3/(1 + t)$ and $FP_r = (x_r, y_r)$, $x_r = y_r = (-1 + t^2 + t^3)/(t^2 + t^3)$. The eigenvalues of $Df_{a,b}(FP_s)$ are $\{t^2, t^3\}$. The invariant curve is $S = \{P_{t,a,b} = 0\}$, with $P$ as in (2.2). This curve $S$ contains $FP_s$ and $FP_r$, and has a cusp at $FP_s$. The point $q$ belongs to $S$, and thus the orbit $f^j q$ for all $j$ until possibly we have $f^j q \in \mathcal{I}$. The 2-cycle and 3-cycle are disjoint from $S$, so the multipliers in (B.1) must satisfy $\mu_2^3 = \mu_3^2$, from which we determine that $\Gamma_1 \subset \mathcal{V}$ is a curve of degree 6.

We use the notation $\delta_\star$ for the real root of $t^3 - t - 1$. Thus $1 \le \lambda_n < \delta_\star$, and the $\lambda_n$ increase to $\delta_\star$ as $n \to \infty$. The intersection of the cubic curve with $\mathbf{RP}^2$ is shown in Figure 2.1. The exceptional curves $\Sigma_\beta$ and $\Sigma_\gamma$ are used as axes, and we have chosen a modification of polar coordinates so that $\Sigma_0$, the line at infinity, appears as the bounding circle of $\mathbf{RP}^2$. The points $FP_{s/o}, e_1, e_2, p, q, fq, f^2q, f^3q$ all belong to $S$, and Figure 2.1 gives their relative positions with respect to the triangle $\Sigma_\beta, \Sigma_\gamma, \Sigma_0$ for all $1 < t < \delta_\star$. Since $t > 1$, the points $f^j q$ for $j \ge 4$ lie on the arc connecting $f^4 q$ and $FP_r$, and $f^j q$ approaches $FP_r$ monotonically along this arc as $j \to \infty$. In case $(a, b)$ belongs to $\mathcal{V}_n$, then $f^n q$ lands on $p$. The relative position of $S_t$ with respect to the axes is stable for $t$ in a large neighborhood of $[1, \delta_\star]$. However, as $t$ increases to $\delta_\star$, the fixed point $FP_r$ moves down to $p$; and for $t > \delta_\star$, $FP_r$ is in the third quadrant. And as $t$ decreases to 1, $f$ approaches the (integrable) map $(a, b) = (-1/4, 0) \in \mathcal{V}_6$. The family $\mathcal{V}_6$ will be discussed in Appendix A. When $0 < t < 1$, the point $FP_s$ becomes attracting, and the relative positions of $q$ and $fq$, etc., are reversed. Figure 2.1 will be useful in explaining the graph shown in Figure 4.1.

$\Gamma_2$*: Line tangent to a quadric.* Next we suppose that $(a, b) = \varphi_2(t)$. We let $S = \{P_{t,a,b} = 0\}$ be the curve in (2.2). In this case, the curve is the union of a line $L = \{t^2 x_0 + t x_1 + x_2 = 0\}$ and a quadric $Q$. The fixed points are $FP_s = (x_s, y_s)$, $x_s = y_s = -t^2/(1 + t)$ and

$FP_r = (x_r, y_r)$, $x_r, y_r = (1+t+t^2)/(t+t^2)$. The eigenvalues of $Df_{a,b}$ at $FP_s$ are $\{-t, -t^2\}$. The 3-cycle and $FP_r$ are disjoint from $S$, so we have $\det(Df_{a,b}FP_r)^3 - \mu_3 = 0$ on $\Gamma_2$, with $\mu_3$ as in (B.1). Extracting an irreducible factor, we find that $\Gamma_2 \subset \mathcal{V}$ is a quartic.

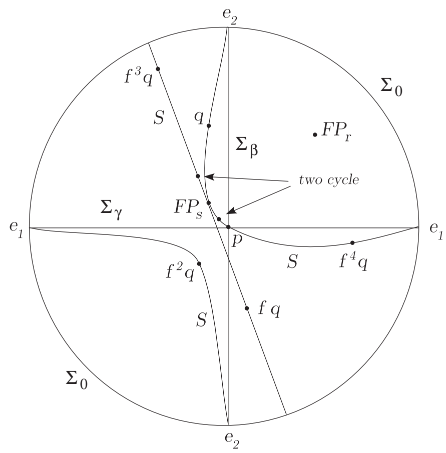


Figure 2.2. Orbit of $q$ for family $\Gamma_2$; $1 < t < \delta_\star$.

Figure 2.2 gives for $\Gamma_2$ the information analogous to Figure 2.1. The principal difference with Figure 2.1 is that $S$ contains an attracting 2-cycle; there is a segment $\sigma$ inside the line connecting $f^3q$ to one of the period-2 points, and there is an arc $\gamma \ni p$ inside the quadric connecting $f^4q$ to the other period-2 point. Thus the points $f^{2j+1}q$ will approach the two-cycle monotonically inside $\sigma$ as $j \to \infty$, and the points $f^{2j}q$ will approach the two-cycle monotonically inside $\gamma$. The picture of $S$ with respect to the triangle $\Sigma_\beta, \Sigma_\gamma, \Sigma_0$ is stable for $t$ in a large neighborhood of $[1, \delta_\star]$. As $t$ increases to $\delta_\star$, one of the points of the 2-cycle moves down to $p$. As $t$ decreases to 1, $q$ moves up (and $f^2q$ moves down) to $e_2 \in \mathcal{I}$, and $fq$ moves down to $\Sigma_0$. The case $t = 1$ is discussed in Appendix A.

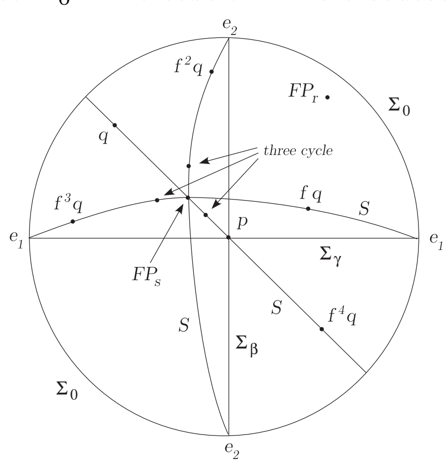


Figure 2.3. Orbit of $q$ for family $\Gamma_3$; $1 < t < \delta_\star$.

$\Gamma_3$*: Three lines passing through a point.* Finally, set $(a, b) = \varphi_3(t)$, and let $S = \{P_{t,a,b} = 0\}$ be given as in (2.2). The fixed points are $FP_s = (x_s, y_s)$, $x_s = y_s = -t$ and $FP_r = (x_r, y_r)$, $x_r = y_r = 1 + t^{-1}$. The invariant set $S$ is the union of three lines $L_1 = \{tx_0 + x_1 = 0\}$, $L_2 = \{tx_0 + x_2 = 0\}$, $L_3 = \{(t+t^2)x_0 + tx_1 + x_2 = 0\}$, all of which pass through $FP_s$.

Further $p, q \in L_3 \to L_2 \to L_1$. The eigenvalues of $Df_{a,b}$ at $FP_s$ are $\{\omega t, \omega^2 t\}$, where $\omega$ is a primitive cube root of unity. The 2-cycle and $FP_r$ are disjoint from $S$, so we have $\det(Df_{a,b}FP_r)^2 - \mu_2 = 0$ on $\Gamma_2$. Extracting an irreducible factor from this equation we see that $\Gamma_3 \subset \mathcal{V}$ is a quadric. Figure 2.3 is analogous to Figures 2.1 and 2.2; the lines $L_1$ and $L_2$ appear curved because of the choice of coordinate system.

**Theorem 2.2.** *Suppose that $n$, $1 \le j \le 3$, and $t$ are given, and suppose that $(a, b) := \varphi_j(t) \notin \mathcal{V}_k$ for any $k < n$. Then the point $(a, b)$ belongs to $\mathcal{V}_n$ if and only if: $j$ divides $n$ and $t$ is a root of $\chi_n$.*

*Proof.* Let us start with the case $j = 3$ and set $(a, b) = \varphi_3(t)$. By the calculation above, we know that $S = L_1 \cup L_2 \cup L_3$ factors into the product of lines, each of which is invariant under $f^3$. $L_3$ contains $FP_s$ and $R = [t^2 : -1 : t - t^3 - t^4]$, which is periodic of period 3. We define $\psi(\zeta) = FP_s + \zeta R$, which gives a parametrization of $L_3$; and the points $\psi(0)$ and $\psi(\infty)$ are fixed under $f^3$. The differential of $f^3$ at $FP_s$ was seen to be $t^3$ times the identity, so we have $f^3(\psi(\zeta)) = \psi(t^3\zeta)$. Now set $\zeta_q := t^2/(1 - t^2 - t^3)$ and $\zeta_p := t/(t^3 - t - 1)$. It follows that $\psi(\zeta_q) = q$ and $\psi(\zeta_p) = p$. If $n = 3k$, then $f^n q = f^{3k} q = p$ can hold if and only if $t^n\zeta_q = t^{3k}\zeta_q = \zeta_p$, or $t^{n+2}/(1-t^2-t^3) = t/(t^3-t-1)$, which is equivalent to $\chi_n(t) = 0$.

Next, suppose that $j = 2$ and let $(a, b) = \varphi_2(t)$. In this case the polynomial $P$ given in (2.2) factors into the product of a line $L$ and a quadric $Q$. $L$ contains $FP_s$ and the point $R = [t + t^2 : t^3 + t^2 - 1 : -t]$, which has period 2. We parametrize $L$ by the map $\psi(\zeta) = FP_s + \zeta R$. Now $f^2$ fixes $FP_s$ and $R$, and the differential of $f^2$ has an eigenvalue $t^2$ in the eigenvector $L$, so we have $f^2\psi(\zeta) = \psi(t^2\zeta)$. Since $p, q \in Q$, we have $fq, f^{-1}p \in L$. We see that $\zeta_q := t^3/(1-t^2-t^3)$ and $\zeta_p := (t^3-t-1)^{-1}$ satisfy $\psi(\zeta_q) = fq$ and $\psi(\zeta_p) = f^{-1}p$. If $n = 2k$, then the condition $f^n q = f^{2k} q = p$ is equivalent to the condition $t^{2n-2}\zeta_q = \zeta_p$, which is equivalent to $\chi_n(t) = 0$.

Finally we consider the case $j = 1$ and set $(a, b) = \varphi_1(t)$. If we substitute these values of $(a, b)$ into the formula (2.2), we obtain a polynomial $P(x)$ which is cubic in $x$ and which has coefficients which are rational in $t$. In order to parametrize $S$ by $\mathbf{C}$, we set $\psi(\zeta) = FP_s + \zeta A + \zeta^2 B + \zeta^3 FP_r$. We may solve for $A = A(t)$ and $B = B(t)$ such that $P(\psi(\zeta)) = 0$ for all $\zeta$. Thus $f$ fixes $\psi(0)$ and $\psi(\infty)$, and $f(\psi(\zeta)) = \psi(t\zeta)$. We set $\zeta_q := t^2/(1 - t^2 - t^3)$ and $\zeta_p := t/(t^3 - t - 1)$. The condition $f^n q = p$ is equivalent to $t^n\zeta_q = \zeta_p$, or $-t^{n+2}/(t^3 + t^2 - 1) = t/(t^3 - t - 1)$, or $\chi_n(t) = 0$. ☐

For each $n$, we let $\psi_n(t)$ denote the minimal polynomial of $\lambda_n$.

**Theorem 2.3.** *Let $t \ne 1$ be a root of $\chi_n$ for $n \ge 7$. Then either $t$ is a root of $\psi_n$, or $t$ is a root of $\chi_j$ for some $0 \le j \le 5$.*

*Proof.* Let $t$ be a root of $\chi_n$. It suffices to show that if $t$ is a root of unity, then it is a root of $\chi_j$ for some $0 \le j \le 5$. First we note that $\chi_6(t) = (t-1)^3(t+1)(t^2+t+1)(t^4+t^3+t^2+t+1)$, and $\chi_7 = (t - 1)\psi_7(t)$. Since every root of $\chi_6$ is a root of $\chi_j$ for some $0 \le j \le 5$, and the Theorem is evidently true for $n = 7$, then by induction it suffices to show that if $t$ is a root of unity, then it is a root of $\chi_j$ for some $0 \le j \le n - 1$. By Theorem 2.2, we see that $\chi_n(t) = 0$ if and only if $t^n\zeta_q(t) = \zeta_p(t)$, where $\zeta_q(t) = t^2/(1 - t^2 - t^3)$ and $\zeta_p(t) = t/(t^3 - t - 1)$. Note that if $t$ is a root of $\chi_n$, then so is $1/t$, and that $\zeta_q(t) = \zeta_p(1/t)$.

Claim 1: We may assume $t^n \neq \pm 1$. Otherwise, from $t^n \zeta_q(t) = \zeta_p(t)$ we have

$$t^2(t^3 - t - 1) \pm t(t^3 + t^2 - 1) = 0.$$

In case we take "+", the roots are also roots of $\chi_0$, and in case we take "−", there are no roots of unity.

Claim 2: If $t^k = 1$ for some $0 \leq k \leq n-1$, then $t$ is a root of $\chi_j$ for some $0 \leq k-1$. As in the proof of Theorem 2.2, the orbit of $\zeta_q$ is $\{\zeta_q, t\zeta_q, \ldots, t^{k-1}\zeta_q\}$. Thus the condition that $t^k = 1$ means that $f^k q = p$, so $\chi_k(t) = 0$.

Claim 3: If $t^k = 1$ for $n+1 \leq k \leq 2n-1$, then $t$ is a root of $\chi_j$ for some $0 \leq j \leq k-n$. Since $t^n \zeta_q(t) = \zeta_p(t)$, we have $t^{k-n}\zeta_p(t) = t^k \zeta_q(t) = \zeta_q(t)$. By our observations above, $\zeta_q(1/t) = (1/t)^{k-n}\zeta_p(1/t)$, and $0 \leq k-n \leq n-1$. It follows that $1/t$ is a root of $\chi_j$ for some $0 \leq j \leq k-n$, and thus $t$, too, is a root of $\chi_j$.

Claim 4: If $\chi_n(t) = 0$, then $t$ is not a primitive $k$-th root of unity for any $k > 2n$. Since $t^n \zeta_q(t) = \zeta_p(t)$ we have $\zeta_q(1/t) = (1/t)^{k-n}\zeta_p(1/t)$, and $k-n > n$. And $1/t$ is also a root of $\chi_n$, and therefore $(1/t)^{k-2n}\zeta_q(1/t) = \zeta_p(1/t)$, and $t$ is a $(k-2n)$-th root of unity, which contradicts our assumptions. □

The following result gives the possibilities for the roots of $\chi_n(x)/\psi_n(x) \in \mathbf{Z}[x]$.

**Theorem 2.4.** *Let $t \neq 1$ be a root of $\chi_n$ with $n \geq 7$. Then $t$ is either a root of $\psi_n$, or $t$ is a root of some $\chi_j$ for $0 \leq j \leq 5$. Specifically, if $t$ is not a root of $\psi_n$, then it is a $k$th root of unity corresponding to one of the following possibilities:*

*(i)* $k = 2$, $t + 1 = 0$, *in which case 2 divides $n$;*
*(ii)* $k = 3$, $t^2 + t + 1 = 0$, *in which case 3 divides $n$;*
*(iii)* $k = 5$, $t^4 + t^3 + t^2 + t + 1 = 0$, *in which case* $n \equiv 1 \bmod 5$;
*(iv)* $k = 8$, $t^4 + 1 = 0$, *in which case* $n \equiv 2 \bmod 8$;
*(v)* $k = 12$, $t^4 - t^2 + 1 = 0$, *in which case* $n \equiv 3 \bmod 12$;
*(vi)* $k = 18$, $t^6 - t^3 + 1 = 0$, *in which case* $n \equiv 4 \bmod 18$;
*(vii)* $k = 30$, $t^8 + t^7 - t^5 - t^4 - t^3 + t + 1 = 0$, *in which case* $n \equiv 5 \bmod 30$.

*Conversely, for each $n \geq 7$ and $k$ satisfying one of the conditions above, there is a corresponding root $t$ of $\chi_n$ which is a $k$th root of unity.*

*Proof.* Recall that $\chi_n(t) = 0$ if and only if $t^n \zeta_q(t) = \zeta_p(t)$ by Theorem 2.2. If $t$ is a $k$-th root of unity, then $k < n$ and $\chi_j(t) = 0$ for some $0 \leq j \leq 5$. In case $j = 0$, $\zeta_p(t) = \zeta_q(t)$, and $(t+1)(t^2+t+1) = 0$. Thus $t^n \zeta_q(t) = \zeta_p(t)$ if and only if $t + 1 = 0$ and 2 divides $n$, or $t^2 + t + 1 = 0$ and 3 divides $n$. Now let us write $k_j = 5, 8, 12, 18, 30$ for $j = 1, 2, 3, 4, 5$, respectively, in case we have $1 \leq j \leq 5$, $t^j \zeta_q(t) = \zeta_p(t)$, and $n \equiv j \bmod k_j$. Thus $t^n \zeta_q(t) = \zeta_p(t)$ if and only if $n \equiv j \bmod k_j$. That is, $t^n \zeta_q(t) = (t^{k_j})^n t^j \zeta_q(t) = t^j \zeta_q(t) = \zeta_p(t)$. □

As a corollary, we see that the number of elements of $\Gamma_j \cap \mathcal{V}_n$ is determined by the number of Galois conjugates of $\lambda_n$.

**Corollary 2.5.** *If $n \geq 7$, and if $1 \leq j \leq 3$ divides $n$, then*

$$\Gamma_j \cap \mathcal{V}_n = \{\varphi_j(t) : t \text{ is a root of } \psi_n\}.$$

*In particular, these sets are nonempty.*

**Theorem 2.6.** *If $n \geq 7$, then every root of $\chi_n$ is simple. Thus the possibilities enumerated in Theorem 2.4 give the irreducible factorization of $\chi_n$.*

*Proof.* If $t$ is a root of $\chi_n$, then either it is a root of $\psi_n$, which is irreducible, or it is one of the roots of unity listed in Theorem 2.4. We have

$$\chi_n'(t) = (n+4)t^{n+3} - (n+2)t^{n+1} - (n+1)t^n + 3t^2 + 2t.$$

Since $\chi_n'(1) = 6 - n$, 1 is a simple root. Now we check all the remaining cases:

(i) 2 divides $n$ : $t + 1 = 0 \Rightarrow \chi_n'(t) = -2 - n \neq 0$.

(ii) 3 divides $n$ : $t^2 + t + 1 = 0 \Rightarrow \chi_n'(t) = 3t^2 - nt + 3 \neq 0$.

(iii) $n \equiv 1 \pmod 5$ : $t^4 + t^3 + t^2 + t + 1 = 0 \Rightarrow \chi_n'(t) = (n+4)t^4 - (n-1)t^2 - (n-1)t \neq 0$.

(iv) $n \equiv 2 \pmod 8$ : $t^4 + 1 = 0 \Rightarrow \chi_n'(t) = -(n+2)t^3 - (n-2)t^2 - (n+2)t \neq 0$.

(v) $n \equiv 3 \pmod{12}$ : $t^4 - t^2 + 1 = 0 \Rightarrow \chi_n'(t) = -(n+1)t^3 - (n-1)t^2 + 2t - 2 \neq 0$.

(vi) $n \equiv 4 \pmod{18}$ : $t^6 - t^3 + 1 = 0 \Rightarrow \chi_n'(t) = -(n+2)t^5 + 3t^4 + 3t^2 - (n+2)t \neq 0$.

(vii) $n \equiv 5 \pmod{30}$ : $t^8 + t^7 - t^5 - t^4 - t^3 + t + 1 = 0 \Rightarrow$
$\chi_n'(t) = (n+4)t^8 - (n+2)t^6 - (n+1)t^5 + 3t^2 + 2t \neq 0$. □

**Example.** The number $n = 26$ corresponds to cases (i), (iii), and (iv), so we see that $\chi_{26} = (t-1)(t+1)(t^4+1)(t^4+t^3+t^2+t+1)\psi_{26}$, so $\psi_{26}$ has degree 20.

§**3. Surfaces without Anti-PluriCanonical Section.** A curve $S$ is said to be a pluri-anticanonical curve if it is the zero set of a section of $\Gamma(\mathcal{X}, (-K_{\mathcal{X}})^{\otimes n})$ for some $n > 0$. We will say that $(\mathcal{X}, f)$ is minimal if whenever $\pi : \mathcal{X} \to \mathcal{X}'$ is a birational morphism mapping $(\mathcal{X}, f)$ to an automorphism $(\mathcal{X}', f')$, then $\pi$ is an isomorphism. Gizatullin conjectured that if $\mathcal{X}$ is a rational surface which has an automorphism $f$ such that $f^*$ has infinite order on $Pic(\mathcal{X})$, then $\mathcal{X}$ should have an anti-canonical curve. Harbourne [H] gave a counterexample to this, but this counterexample is not minimal, and $f$ has zero entropy.

**Proposition 3.1.** *Let $\mathcal{X}$ be a rational surface with an automorphism $f$. Suppose that $\mathcal{X}$ admits a pluri-anticanonical section. Then there is an $f$-invariant curve.*

*Proof.* Suppose there is a pluri-anticanonical section. Then $\Gamma(\mathcal{X}, (-K_{\mathcal{X}})^{\otimes n})$ is a nontrivial finite dimensional vector space for some $n > 0$, and $f$ induces a linear action on this space. Let $\eta$ denote an eigenvalue of this action. Since $\mathcal{X}$ is a rational surface, $S = \{\eta = 0\}$ is a nontrivial curve, which must be invariant under $f$. □

The following answers a question raised in [M2, §12].

**Theorem 3.2.** *There is a rational surface $\mathcal{X}$ and an automorphism $f$ of $\mathcal{X}$ with positive entropy such that $(\mathcal{X}, f)$ is minimal, but there is no $f$-invariant curve. In particular, there is no pluri-canonical section.*

*Proof.* We consider $(a, b) \in \mathcal{V}_{11}$. Suppose that $(\mathcal{X}_{a,b}, f_{a,b})$ has an invariant curve $S$. Then by Theorem 1.2, $S$ must be a cubic. By Theorem 2.2., then we must have $(a, b) \in \Gamma_1$. That is, $(a, b) = \varphi_1(t)$ for some $t$. By Theorem 2.4, $t$ is a root of the minimal polynomial $\psi_{11}$. By Theorem 2.6, $\psi_{11}(t) = \chi_{11}(t)/((t-1)(t^4+t^3+t^2+t+1))$ has degree 10, so $\mathcal{V}_{11} \cap \Gamma_1$ contains

10 elements. However, there are 12 elements in $\mathcal{V}_{11} - \Gamma_1$; a specific example is given in Appendix B. Each of these gives an automorphism $(\mathcal{X}_{a,b}, f_{a,b})$ with entropy $\log \lambda_{11} > 0$ and with no invariant curve. Since $\mathcal{X}_{11}$ was obtained by starting with $\mathbf{P}^2$ and blowing up the minimal set necessary to remove singularities, it is evident that $(\mathcal{X}_{a,b}, f_{a,b})$ is minimal. The nonexistence of an pluri-anticanonical section follows from Proposition 3.1. □

**Remark.** By Proposition B.1 we cannot take $n \le 10$ the proof of Theorem 3.2.

§**4. Rotation (Siegel) Domains.** Given an automorphism $f$ of a compact surface $\mathcal{X}$, we define the Fatou set $\mathcal{F}$ to be the set of normality of the iterates $\{f^n : n \ge 0\}$. Let $\mathcal{D}$ be an invariant component of $\mathcal{F}$. We say that $\mathcal{D}$ is a rotation domain if $f_{\mathcal{D}}$ is conjugate to a linear rotation (cf. [BS1] and [FS]). In this case, the normal limits of $f^n|_{\mathcal{D}}$ generate a compact abelian group. In our case, the map $f$ does not have finite order, so the iterates generate a torus $\mathbf{T}^d$, with $d = 1$ or $d = 2$. We say that $d$ is the rank of $\mathcal{D}$. The rank is equal to the dimension of the closure of a generic orbit of a point of $\mathcal{D}$. McMullen [M2] showed that if $n \ge 8$ there are rank 2 rotation domains centered at $FP_r$ in the family $\Gamma_2 \cap \mathcal{V}_n$ (if 2 divides $n$) and $\Gamma_3 \cap \mathcal{V}_n$ (if 3 divides $n$).

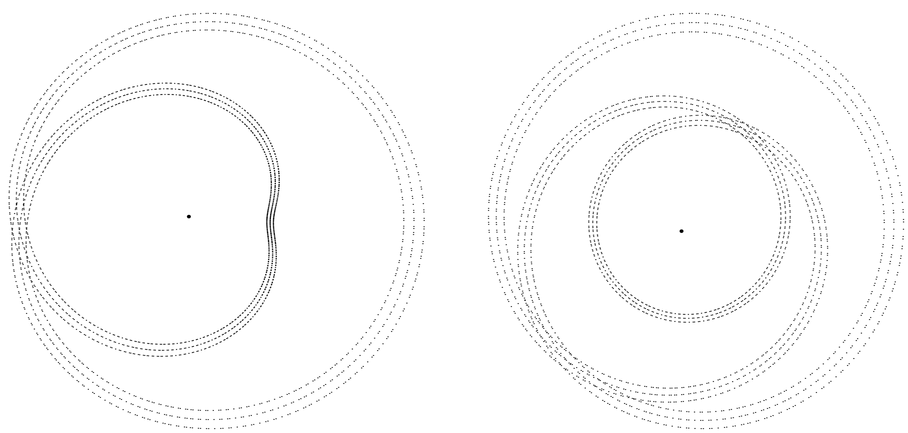

Figure 4.1. Orbits of three points in the rank 1 rotation domain containing $FP_s$; $(a,b) \in \mathcal{V}_7 \cap \Gamma_1$. Two projections.

**Theorem 4.1.** *Suppose that $n \ge 7$, $j$ divides $n$, and $(a,b) \in \Gamma_j \cap \mathcal{V}_n$. That is, $(a,b) = \varphi_j(t)$ for some $t \in \mathbf{C}$. If $t \ne \lambda_n, \lambda_n^{-1}$ is a Galois conjugate of $\lambda_n$, then $f_{a,b}$ has a rotation domain of rank 1 centered at $FP_s$.*

*Proof.* There are three cases. We saw in §1 that the eigenvalues of $Df_{a,b}$ at $FP_s$ are $\{t^2, t^3\}$ if $j = 1$; they are $\{-t, -t^2\}$ if $j = 2$ and $\{\omega t, \omega^2 t\}$ if $j = 3$. Since $\lambda_n$ is a Salem number, the Galois conjugate $t$ has modulus 1. Since $t$ is not a root of unity, it satisfies the Diophantine condition

$$|1 - t^k| \ge C_0 k^{-\nu} \tag{4.1}$$

for some $C_0, \nu > 0$ and all $k \ge 2$. This is a classical result in number theory. A more recent proof (of a more general result) is given in Theorem 1 of [B]. We claim now that if $\eta_1$ and $\eta_2$ are the eigenvalues of $Df_{a,b}$ at $FP_s$, then for each $m = 1, 2$, we have

$$|\eta_m - \eta_1^{j_1}\eta_2^{j_2}| \ge C_0(j_1 + j_2)^{-\nu} \tag{4.2}$$

for some $C_0, \eta > 0$ and all $j_1 + j_2 \geq 2$. There are three cases to check: $\Gamma_j$, $j = 1, 2, 3$. In case $j = 1$, we have that $\eta_m - \eta_1^{j_1}\eta_2^{j_2}$ is equal to either $t^2 - t^{2j_1+3j_2} = t^2(1 - t^{2(j_1-1)+3j_2})$ or $t^3 - t^{2j_1+3j_2} = t^3(1 - t^{2j_1+3(j_2-1)})$. Since $j_1 + j_2 > 1$, we see that (4.2) is a consequence of (4.1). In the case $j = 2$, $Df^2$ has eigenvalues $\{t^2, t^4\}$, and in the case $j = 3$, $Df^3$ has eigenvalues $\{t^3, t^3\}$. In both of these cases we repeat the argument of the case $j = 1$. It then follows from Zehnder [Z2] that $f_{a,b}$ is holomorphically conjugate to the linear map $L = \mathrm{diag}(\eta_1, \eta_2)$ in a neighborhood of $FP_s$. □

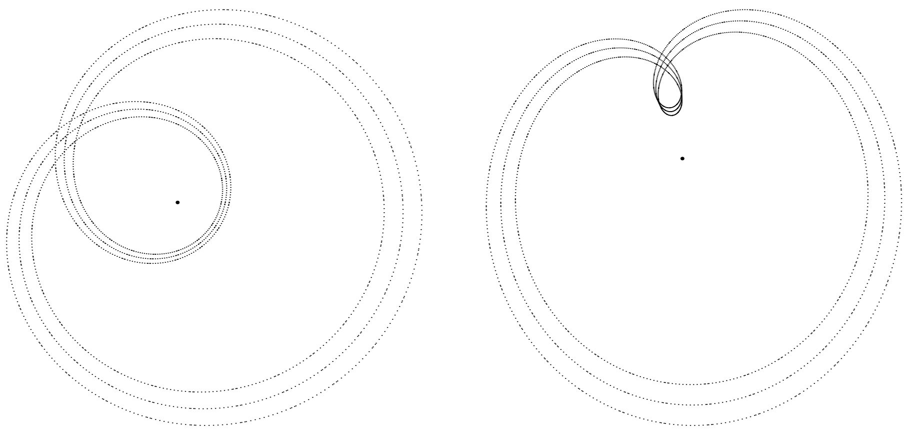

Figure 4.2. $f^2$-orbits of three points in the rank 1 rotation domain containing $FP_s$; $(a, b) \in \mathcal{V}_8 \cap \Gamma_2$. Two projections.

**Remark.** Now let us discuss the other fixed point. Suppose that $(a, b) \in \Gamma_1 \cap \mathcal{V}_n$ and $\{\eta_1, \eta_2\}$ are the multipliers at $FP_r$. As was noted in the proof above, $t$ satisfies (4.1), and so by Corollary B.5, both $\eta_1$ and $\eta_2$ satisfy (4.1). On the other hand, the resonance given by Theorem B.3 means that they do not satisfy (4.2), and thus we cannot conclude directly that $f$ can be linearized in a neighborhood of $FP_r$. However, by Pöschel [P], there are holomorphic Siegel disks (of complex dimension one) $s_j : \{|\zeta| < r\} \to \mathcal{X}_{a,b}$, $j = 1, 2$, with the property that $s_j'(0)$ is the $\eta_j$ eigenvector, and $f(s_j(\zeta)) = s_j(\eta_j \zeta)$. We note that one of these Siegel disks will lie in the invariant cubic itself. And by Theorem B.4 there are similar resonances between the multipliers for the 2- and 3-cycles, and thus similar Siegel disks, in the cases $\Gamma_2 \cap \mathcal{V}_n$ and $\Gamma_3 \cap \mathcal{V}_n$ respectively.

**Remark.** For each $n \geq 7$ and each divisor $1 \leq j \leq 3$ of $n$, the only values of $(a, b) \in \mathcal{V}_n \cap \Gamma_j$ to which Theorem 4.1 does not apply are the two values $\varphi_j(\lambda_n)$ and $\varphi_j(\lambda_n^{-1})$. For all the other maps in $\mathcal{V}_n \cap \Gamma$, the Siegel domain $\mathcal{D} \ni FP_s$ is a component of both Fatou sets $\mathcal{F}(f)$ and $\mathcal{F}(f^{-1})$. For instance, if $j = 1$, then $f$ is conjugate on $\mathcal{D}$ to the linear map $(z, w) \mapsto (t^2 z, t^3 w)$. Thus, in the linearizing coordinate, the orbit of a point of $\mathcal{D}$ will be dense in the curve $\{|z| = 1, w^2 = cz^3\}$, for some $r$ and $c$. In particular, the closure of the orbit bounds an invariant (singular) complex disk. Three such orbits are shown in Figure 4.1. Similarly, if $j = 2$, then $f^2$ is conjugate on $\mathcal{D}$ to $(z, w) \mapsto (t^2 z, t^4 w)$. Thus the $f^2$-orbit of a point $\mathcal{D}$, shows in Figure 4.2, will be dense in the boundary of $\{|z| = r, w = cz^2\}$. The whole $f$-orbit will be (dense in) the union of two such curves.

**Corollary 4.2.** *If $n$, $j$, and $(a, b)$ are as in Theorem 4.1, and if $j = 2$ or 3, then $f_{a,b}$ has (at least) two rotation domains.*

*Proof.* Theorem 4.1 gives a rank 1 rotation domain centered at $FP_s$. If $j = 2$ or 3, then $FP_r$ is not contained in the invariant cubic, and McMullen [M2] gives a rank 2 rotation domain centered at $FP_r$. □

§**5. Real Mappings of Maximal Entropy.** Here we consider real parameters $(a, b) \in \mathbf{R}^2 \cap \mathcal{V}_n$ for $n \geq 7$. Given such $(a, b)$, we let $\mathcal{X}_R$ denote the closure of $\mathbf{R}^2$ inside $\mathcal{X}_{a,b}$. We let $\lambda_n > 1$ be the largest root of $\chi_n$, and for $1 \leq j \leq 3$, we let $f_{j,R}$ denote the automorphism of $\mathcal{X}_R$ obtained by restricting $f_{a,b}$ to $\mathcal{X}_R$, with $(a, b) = \varphi_j(\lambda_n)$.

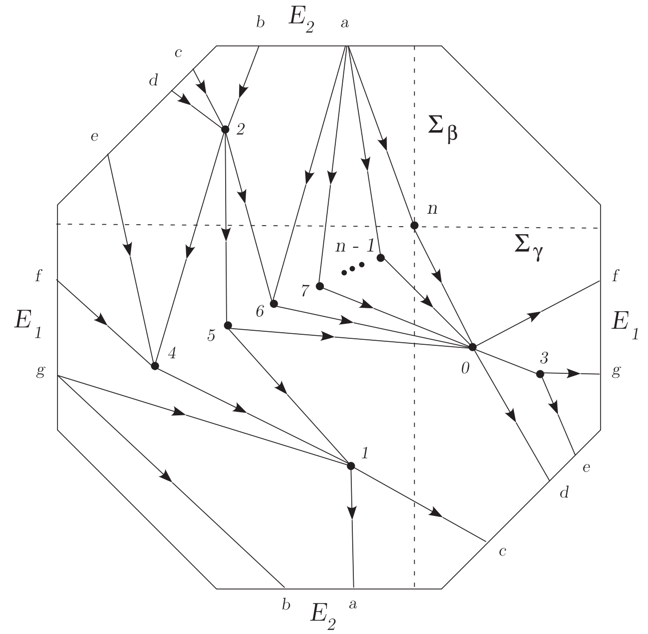


Figure 5.1. Graph $\mathcal{G}_1$; Invariant homology class for family $\Gamma_1$.

**Theorem 5.1.** *There is a homology class $\eta \in H_1(\mathcal{X}_R)$ such that $f_{1,R*}\eta = -\lambda_n\eta$. In particular, $f_{1,R}$ has entropy $\log \lambda_n$.*

*Proof.* We use an octagon in Figure 5.1 to represent $\mathcal{X}_R$. Namely, we start with the real projective plane $\mathbf{RP}^2$; we identify antipodal points in the four "slanted" sides. The horizontal and vertical pairs of sides of the octagon represent the blowup fibers over the points $e_1$ and $e_2$. These are labeled $E_1$ and $E_2$; the letters along the boundary indicate the identifications. (Since we are in a blowup fiber, the identification is no longer "antipodal.") Further, the points $f^j q$ (written "$j$") $0 \leq j \leq n$, are blown up, although we do not draw the blowup fibers explicitly. To see the relative positions of "$j$" with respect to the triangle $\Sigma_\beta, \Sigma_\gamma, \Sigma_0$, consult Figure 2.1. The 1-chains of the homology class $\eta$ are represented by the directed graph $\mathcal{G}_1$ inside the manifold $\mathcal{X}_R$. If we project $\mathcal{X}_R$ down to the projective plane, then all of the incoming arrows at a center of blowup "$j$" will be tangent to each other, as well as the outgoing arrows.

In order to specify the homology class $\eta$, we need to assign real weights to each edge of the graph. By "51" we denote the edge connecting "5" and "1"; and "170 = $1a70$" denotes the segment starting at "1", passing through "$a$", continuing through "7", and ending at "0". Abusing notation, also write "51", etc., to denote the weight of the edge, as well as the edge itself. We determine the weights by mapping $\eta$ forward. We find that, upon mapping by $f$, the orientations of all arcs are reversed. Let us describe how to do this. Consider the arc "34"="3$e$4". The point "$e$" belongs to $\Sigma_0$, and so it maps to $E_1$. Thus "34" is mapped to something starting at "4", passing through $E_1$, and then continuing to "5". Thus we see that "34" is mapped (up to homotopy) to "4f05". Thus, the image of "34" covers "04" and "05".

Inspection shows that no other arc maps across "04", so we write "04 → 34" to indicate that the weight of side "04" in $f_*\eta$ is equal to the weight of "34". Inspecting the images of all the arcs, we find that "24" also maps across "05", so we write "05 → 24 + 34" to indicate how the weights transform as we push $\mathcal{G}_1$ forward. Looking at all possible arcs, we write the transformation $\eta \mapsto f_*\eta$ as follows:

$$\begin{array}{l} 02 \to 16 + 170 + \cdots + 1(n-1)0, \quad 03 \to 24 + 25 + 26, \quad 04 \to 34, \\ 05 \to 24 + 34, \quad 06 \to 25, \quad 12 \to 1n0, \quad 13 \to 02, \quad 14 \to 03, \\ 15 \to 04, \quad 16 \to 05, \quad 170 \to 06, \quad 1k0 \to 1(k-1)0, \ 7 < k \le n, \\ 23 \to 12, \quad 24 \to 13, \quad 25 \to 14, \quad 26 \to 15, \quad 34 \to 23, \end{array} \tag{5.1}$$

The formula (5.1) defines a linear transformation on the space of coefficients of the 1-chains defining $\eta$. The spectral radius of the transformation (5.1) is computed in Appendix C, where we find that it is $\lambda_n$. Now let $w$ denote the eigenvector of weights corresponding to the eigenvalue $\lambda_n$. It follows that if we assign these weights to $\eta$, then by construction we have $f_{1,R*}\eta = -\lambda_n\eta$, and $\eta$ is closed. □

**Remark.** Let us compare with the situation for real Hénon maps. In [BLS] it was shown that a real Hénon map has maximal entropy if and only if all periodic points are real. On the other hand, if $(a, b) = \varphi_1(t)$, $1 \le t \le 2$, the (unique) 2-cycle of the map $f_{a,b}$ is non-real. This includes all the maps discussed in Theorem 5.1, since all values of $t = \lambda_n$ are in this interval.

**Theorem 5.2.** *The maps $f_{2,R}$ (if $n$ is even) and $f_{3,R}$ (if $n$ is divisible by 3) have entropy* $\log \lambda_n$.

*Proof.* Since the entropy of the complex map $f$ on $\mathcal{X}$ is $\log \lambda_n$, the entropies of $f_{2,R}$ and $f_{3,R}$ are bounded above by $\log \lambda_n$. In order to show that equality holds for the entropy of the real maps, it suffices by Yomdin's Theorem [Y] (see also [G]) to show that $f_{j,R}$ expands lengths by an asymptotic factor of $\lambda_n$. We will do this by producing graphs $\mathcal{G}_2$ and $\mathcal{G}_3$ on which $f$ has this expansion factor. We start with the case $n = 2k$; the graph $\mathcal{G}_2$ is shown in Figure 5.2, which should be compared with Figure 2.2. Note that as drawn in Figure 5.2, $\mathcal{G}_2$ looks something like a train track in order to show how it is to be lifted to a graph in $\mathcal{X}_R$. We use the notation $01 = 0d1$ for the edge in $\mathcal{G}_2$ connecting "0" to "1" by passing through $d$. In this case, the notation already defines the edge uniquely; we have added the $d$ by way of explanation. Now we discuss how these arcs are mapped. The arc 01 crosses $\Sigma_\beta$ and then $E_2 \ni d$ before continuing to "1". Since $\Sigma_\beta$ is mapped to $E_2$ and $E_2$ is mapped to $\Sigma_0$, the image of 01 will start at "1" and cross $E_2$ and then $\Sigma_0$ before reaching "2". Up to homotopy, we may slide the intersection points in $E_2$ and $\Sigma_0$ over to a point $g \in E_2 \cap \Sigma_0$. Thus, up to homotopy, $f$ maps the edge 01 in $\mathcal{G}_2$ to the edge $1g2$.

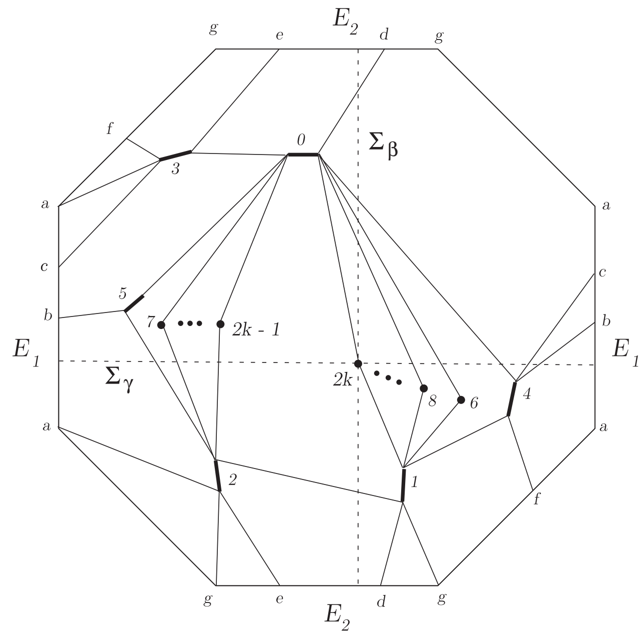


Figure 5.2. Invariant graph $\mathcal{G}_2$: $n = 2k$.

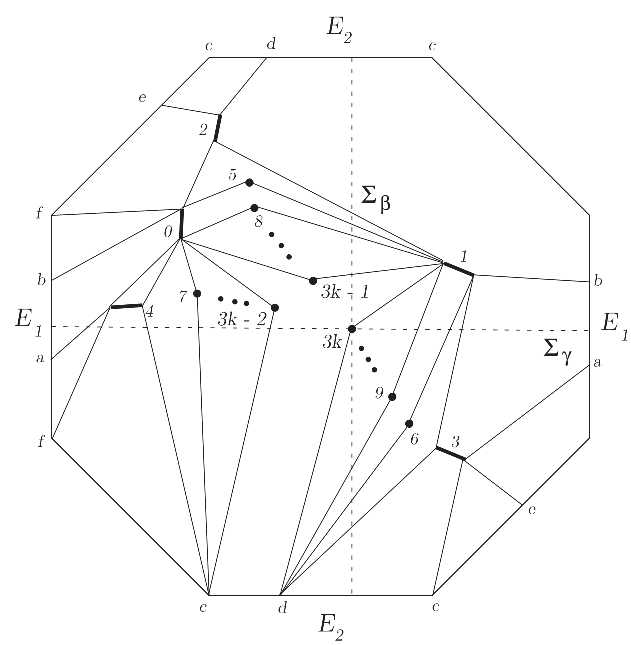


Figure 5.3. Invariant graph $\mathcal{G}_3$: $n = 3k$.

Similarly, we see that the arcs 04, 06, …, $0(2k-1)$ all cross $\Sigma_\beta$ and then $\Sigma_\gamma$. Thus the images of all these arcs will start at 1, pass through $E_2$ at $d$, then 0, and continue to the respective endpoints 5, 7, …, $(2k-1)$. Since the images of all these arcs, up to homotopy, contain the edge 01, the transformation of weights in the graph is given by the first entry of (5.2), and the whole transformation is given by the rest of (5.2):

$$\begin{aligned}&01 \to 04 + 061 + 081 + \cdots + 0(2k-1)1, \quad 03 \to 25 + 072 + 092 + \cdots + 0(2k-1)2\\&04 \to 3c4, \quad 061 \to 25 + 45, \quad 12 \to 0(2k)1, \quad 1g2 \to 01, \quad 14 \to 03\\&0(2j)1 \to 0(2j-1)2,\ j = 4,5,\ldots,k, \quad 0(2j+1)2 \to 0(2j)1, \ \ j = 3,4,\ldots,k-1\\&2a3 \to 12, \quad 2e3 \to 1g2, \quad 25 \to 14, \quad 3c4 \to 2e3, \quad 34 \to 2a3, \quad 45 \to 34.\end{aligned} \tag{5.2}$$

The characteristic polynomial for the transformation defined in (5.2) is computed in the Appendix, and the largest eigenvalue of (5.2) is $\lambda_n$, so $f_{2,R}$ has the desired expansion.

The case $n = 3k$ is similar. The graph $\mathcal{G}_3$ is given in Figure 5.3. Up to homotopy, $f_{3,R}$ maps the graph $\mathcal{G}_3$ to itself according to:

$$\begin{aligned}&01 \to 04, \quad 02 \to 13 + 162 + 192 + \cdots + 1(3k-3)2,\\&04 \to 3a4 + 073 + 0(10)3 + \cdots + 0(3k-2)3, \quad 051 \to 04 + 3c4 + 3a4,\\&0(3j-1)1 \to 0(3j-2)3,\ j = 3,4,\ldots,k, \quad 1(3j)2 \to 0(3j-1)1,\ j = 2,3,\ldots,k,\\&0(3j+1)3 \to 1(3j)2,\ j = 2,3,\ldots,k-1,\\&12 \to 01, \quad 13 \to 02, \quad 23 \to 1(3k)2, \quad 2d3 \to 12, \quad 3a4 \to 23, \quad 3c4 \to 2d3.\end{aligned} \tag{5.3}$$

The linear transformation corresponding to (5.3) is shown in Appendix C to have spectral radius equal to $\lambda_n$, so $f_{3,R}$ has entropy $\log \lambda_n$. □

**§6. Proof of the Main Theorem.** The Main Theorem is a consequence of results we have proved already. Let $f_{a,b}$ be of the form (0.1). By Proposition B.1, we may suppose that $n \geq 11$. Thus if $f$ has an invariant curve, then by Theorem 1.2, it has an invariant cubic, which is given explicitly by Theorem 2.1. Further, by Theorem 2.2, we must have $(a,b) = \varphi_j(t)$ for some $j$ dividing $n$, and a value $t \in \mathbf{C}$ which is a root of $\chi_n$. By Theorem 2.4, $t$ cannot be a root of unity. Thus it is a Galois conjugate of $\lambda_n$. The Galois conjugates of $\lambda_n$ are of two forms: either $t$ is equal to $\lambda_n$ or $\lambda_n^{-1}$, or $t$ has modulus equal to 1.

In the first case, $(a,b) \in \mathcal{V} \cap \mathbf{R}^2$, and thus $f$ is a real mapping. The three possibilities are $(a,b) \in \mathcal{V}_n \cap \Gamma_j$, $j = 1,2,3$, and these are treated in §5. In all cases, we find that the entropy of the real mapping $f_{R,j}$ has entropy equal to $\log \lambda_n$. By Cantat [C2], there is a unique measure $\mu$ of maximal entropy for the complex mapping. Since $f_{R,j}$ has a measure $\nu$ of entropy $\log \lambda_n$, it follows that $\mu = \nu$, and thus $\mu$ is supported on the real points. On the other hand, we know that $\mu$ is disjoint from the Fatou sets of $f$ and $f^{-1}$. McMullen [M2] has shown that the complement of one of the Fatou sets $\mathcal{F}(f)$ or $\mathcal{F}(f^{-1})$ has zero volume. The same argument shows that the complement inside $\mathbf{R}^2$ has zero area. Thus the support of $\mu$ has zero planar area.

The other possibility is that $t$ has modulus 1. In this case, the Main Theorem is a consequence Theorem 4.1. □

**Appendix A. Varieties $\mathcal{V}_j$ and $\Gamma$ for $0 \leq j \leq 6$.** The sets $\mathcal{V}_j$, $0 \leq j \leq 6$ are enumerated in [BK]. We note that $\mathcal{V}_0 = (0,0) \subset \Gamma_2 \cap \Gamma_3$, $\mathcal{V}_1 = (1,0) \subset \Gamma_1$, $\mathcal{V}_2 \subset \Gamma_1 \cap \Gamma_2$, $\mathcal{V}_3 \subset \Gamma_1 \cap \Gamma_3$, and $(\mathcal{V}_4 \cup \mathcal{V}_5) \cap \Gamma = \emptyset$.

Each of the mappings in $\mathcal{V}_6$ has an invariant pencil of cubics; any of these cubics, including nodal cubics and elliptic curves, can be used to synthesize the map, following [M2, §7]. There are two cases: the set $\mathcal{V}_6 \cap \{b \neq 0\}$ (consisting of four points) is contained in $\Gamma_2 \cap \Gamma_3$. The other case, $\mathcal{V}_6 \cap \{b = 0\} = \{(a, 0) : a \neq 0, 1\}$, differs from the cases $\mathcal{V}_n$, $n \neq 0, 1, 6$, because the manifold $\mathcal{X}_{a,0}$ is constructed by iterated blowups ($f^4 q \in E_1$ and $f^2 q \in E_2$, see [BK, Figure 6.2]).

The invariant function $r(x, y) = (x + y + a)(x + 1)(y + 1)/(xy)$ for $f_{a,0}$, which defines the invariant pencil, was found by Lyness [L] (see also [KLR], [KL], [BC] and [Z1]). We briefly describe the behavior of $f_{a,0}$. By $M_\kappa = \{r = \kappa\}$ we denote the level set of $r$ inside $\mathcal{X}_{a,0}$. The curve $M_\infty$ consists of an invariant 5-cycle of curves with self-intersection $-2$:

$$\Sigma_\beta = \{x = 0\} \mapsto E_2 \mapsto \Sigma_0 \mapsto E_1 \mapsto \Sigma_B = \{y = 0\} \mapsto \Sigma_\beta.$$

The restriction of $f^5$ to any of these curves is a linear (fractional) transformation, with multipliers $\{a, a^{-1}\}$ at the fixed points. $M_0$ consists of a 3-cycle of curves with self-intersection $-1$: $\{y + 1 = 0\} \mapsto \{x + 1 = 0\} \mapsto \{x + y + a = 0\}$. The restriction of $f^3$ to any of these lines is linear (fractional) with multipliers $\{a - 1, (a - 1)^{-1}\}$ at the fixed points.

**Theorem A.1.** *Suppose that* $a \notin \{-\frac{1}{4}, 0, \frac{3}{4}, 1, 2\}$, *and* $\kappa \neq 0, \infty$. *If* $M_\kappa$ *contains no fixed point, then* $M_\kappa$ *is a nonsingular elliptic curve, and* $f$ *acts as translation on* $M_\kappa$. *If* $M_\kappa$ *contains a fixed point* $p$, *then* $M_\kappa$ *has a node at* $p$. *If we uniformize* $s : \hat{\mathbf{C}} \to M_\kappa$ *so that* $s(0) = s(\infty) = p$, *then* $f|_{M_\kappa}$ *is conjugate to* $\zeta \mapsto \alpha\zeta$ *for some* $\alpha \in \mathbf{C}^*$.

The intersection $\Gamma_j \cap \{b = 0\} \cap \mathcal{V}_6$ is given by $(-\frac{1}{4}, 0)$, $(\frac{3}{4}, 0)$, or $(0, 2)$, if $j = 1, 2$, or 3, respectively.

**Theorem A.2.** *Suppose that* $a = -\frac{1}{4}, \frac{3}{4}$, *or 2. Then the conclusions of Theorem A.1 hold, with the following exception. If* $FP_s \in M_\kappa$, *then* $M_\kappa$ *is a cubic which has a cusp at* $FP_s$, *or is a line and a quadratic tangent at* $FP_s$, *or consists of three lines passing through* $FP_s$. *If we uniformize a component* $s : \hat{\mathbf{C}} \to M_\kappa$ *such that* $s(\infty) = FP_s$, *then* $f^j|_{M_\kappa}$ *is conjugate to* $\zeta \mapsto \zeta + 1$, *where* $j$ *is chosen so that* $(a, 0) \in \Gamma_j$.

**Appendix B. Varieties $\mathcal{V}_n$ and $\Gamma$ for $n \geq 7$.** We may define the domains $\mathcal{V}_n$ explicitly by starting with the equation $f^n_{a,b}(-a, 0) + (b, a) = 0$ and clearing denominators to convert it to a pair of polynomial equations with integer coefficients in the variables $a$ and $b$. Thus we have a pair of polynomial equations whose solutions contain $\mathcal{V}_n$. Factoring and taking resultants, we may obtain an upper estimate on the number of elements in $\mathcal{V}_n$. In this way, we find that $\#\mathcal{V}_7 \leq 10$. On the other hand, by Theorem 2.6, $\chi_7(x) = (x - 1)\psi_7(x)$, and $\psi_7$ has degree 10. So by Theorem 2.5, $\#(\mathcal{V}_7 \cap \Gamma_1) = 10$, and we conclude that $\mathcal{V}_7 \subset \Gamma_1$. Arguing in this manner, we obtain

**Proposition B.1.** $\mathcal{V}_7 \subset \Gamma_1$, $\mathcal{V}_8 \cup \mathcal{V}_{10} \subset \Gamma_1 \cup \Gamma_2$, *and* $\mathcal{V}_9 \subset \Gamma_1 \cup \Gamma_3$.

**Examples.** When $n > 10$, the sets $\mathcal{V}_n$ exhibit quite a number of maps without invariant curves. By Theorems 2.4 and 2.6, $\Gamma_1 \cap \mathcal{V}_{11} = \Gamma \cap \mathcal{V}_{11}$ contains 10 elements. Using resultants, we find that $\mathcal{V}_{11}$ contains 22 elements, and so $\#\mathcal{V}_{11} - \Gamma = 12$. For instance, there is a parameter $(a,b) \approx (.206286 - .00427394i, -.00802592 + .604835i) \in \mathcal{V}_{11} - \Gamma$. All periodic points of $f_{a,b}$ with period $\le 7$ are saddles; by the Main Theorem this behavior is different from what happens with elements of $\Gamma \cap \mathcal{V}_n$.

Similarly, $\mathcal{V}_{12}$ contains 60 elements: $\#\Gamma_j \cap \mathcal{V}_{12} = 12$ for $j = 1,2,3$, and $\#\mathcal{V}_{12} - \Gamma = 24$. There is a parameter $(a,b) \approx (.586092 + .739242i, .061427 - .940666i) \in \mathcal{V}_{12}$ for which $f_{a,b}$ has both an attracting fixed point and a repelling 5-cycle; such behavior can not come from an element of $\Gamma \cap \mathcal{V}_{12}$.

We note that the parameter values $(\bar{a}, \bar{b})$, $(a-b, -b)$, and $(\bar{a} - \bar{b}, -\bar{b})$, corresponding to complex conjugate of $f_{a,b}$ and taking inverse, or both, also belong to $\mathcal{V}_n$. Thus each of the examples above actually corresponds to four parameter values.

A map $f_{a,b}$ has a unique 2-cycle and a unique 3-cycle. For $\ell = 2, 3$, we let $J_\ell$ denote the product of the Jacobian matrix around the $\ell$-cycle.

**Theorem B.2.** *For $\ell = 2,3$, the determinant $\mu_\ell$ of $J_\ell$ is given by*

$$\mu_2 = \frac{a-b-1}{2b^2+a-1}, \quad \mu_3 = \frac{1+b+b^2-a-ab}{1-a-ab} \tag{B.1}$$

*and the trace $\tau_\ell$ is given by*

$$\tau_2 = \frac{3-2a+b-b^2}{2b^2+a-1}, \quad \tau_3 = \frac{2+a^2+b+2b^2-b^3+b^4+a(-2-b+2b^2)}{-1+a-ab}. \tag{B.2}$$

*Proof.* The proof of the 2-cycle case is simpler and omitted. We may identify a 3-cycle with a triple of numbers $z_1, z_2, z_3$:

$$\zeta_1 = (z_1, z_2) \mapsto \zeta_2 = (z_2, z_3 = \frac{a+z_2}{b+z_1}) \mapsto \zeta_3 = (z_3, z_1 = \frac{a+z_3}{b+z_2}) \mapsto \zeta_1 = (z_1, z_2 = \frac{a+z_1}{b+z_3}).$$

Substituting into this 3-cycle, we find that $z_1, z_2, z_3$ are the three roots of

$$P_3(z) = z^3 + (1+a+b+b^2)z^2 + (b^3+ab+2a-1)z - 1 + a - b + ab - b^2.$$

It follows that

$$\begin{aligned} z_1 + z_2 + z_3 &= -(1+a+b+b^2) \\ z_1z_2 + z_1z_3 + z_2z_3 &= -1+2a+ab+b^3 \\ z_1z_2z_3 &= 1-a+b-ab+b^2 \end{aligned} \tag{B.3}$$

Since

$$Df_{a,b}(\zeta_1 = (z_1,z_2)) = \begin{pmatrix} 0 & 1 \\ -\frac{a+z_2}{(b+z_1)^2} & \frac{1}{b+z_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-z_3}{b+z_1} & \frac{1}{b+z_1} \end{pmatrix},$$

the determinant of $Df_{a,b}(\zeta_1) = z_3/(b+z_1)$, and therefore

$$\mu_3 = \frac{z_2}{b+z_3}\frac{z_1}{b+z_2}\frac{z_3}{b+z_1} = \frac{z_1z_2z_3}{(b+z_3)(b+z_2)(b+z_1)}.$$

Using equations (B.3) we see that $(b+z_1)(b+z_2)(b+z_3) = 1-a+ab$ so $\mu_3$ has the form given in (B.1).

Similarly, we compute

$$Tr(J_3) = -\frac{-1+b(z_1+z_2+z_3)+z_1^2+z_2^2+z_3^2}{(b+z_1)(b+z_2)(b+z_3)}.$$

Using (B.3) again, we find that $\tau_3$ is given in the form (B.2). ∎

A computation shows the following:

**Theorem B.3.** *Suppose that* $(a,b) = \varphi_1(t) \in \mathcal{V}_n \cap \Gamma_1$, $n \geq 7$. *Then the eigenvalues of* $Df$ *at* $FP_r$ *are given by* $\{\eta_1 = 1/t, \eta_2 = -(t^3+t^2-1)/(t^4-t^2-t)\}$, *where* $t$ *is a root of* $\psi_n$. *Further,* $\eta_1^n \eta_2 = 1$.

In the previous theorem, the fixed point $FP_r$ is contained in the invariant curve. If $\ell$ divides $n$, for $\ell = 2$ or 3, then the $\ell$-cycle is disjoint from the invariant curve. Thus we have:

**Theorem B.4.** *Suppose* $\ell = 2$ *or* 3 *and* $n = k\ell \geq 7$. *If* $(a,b) = \varphi_\ell(t) \in \mathcal{V}_n \cap \Gamma_\ell$, *then the eigenvalues of the* $\ell$*-cycle are* $\{\eta_1 = t^{-\ell}, \eta_2 = -t^{\ell-1}(t^3+t^2-1)/(t^3-t-1)\}$. *Further,* $\eta_1^{n+1}\eta_2 = 1$.

**Corollary B.5.** *If* $t = \lambda_n$ *or* $\lambda_n^{-1}$, *then the cycles discussed in Theorems B.3 and B.4 are saddles. If* $t$ *has modulus 1, then the multipliers over these cycles have modulus 1 but are not roots of unity.*

## Appendix C. Computation of Characteristic Polynomials.

**Theorem C.1.** *If* $\chi_n$ *is as in (0.3), and* $n \geq 7$, *then*
(i) *The characteristic polynomial for (5.1) is* $(x^7+1)\chi_n(x)/(x^2-1)$;
(ii) *The characteristic polynomial for (5.2) is* $(x^5-1)\chi_{2k}(x)/(x^2-1)$;
(iii) *The characteristic polynomial for (5.3) is* $(x^4-1)\chi_{3k}(x)/(x^3-1)$.

*Proof.* We start with case $(i)$. Since the case $n = 7$ is easily checked directly, it suffices to prove $(i)$ for $n \geq 8$. Let us use the ordered basis:

$$\{12, 23, 34, 04, 15, 26, 03, 14, 25, 05, 16, 02, 13, 24, 06, 170, 180, \ldots, 1n0\},$$

and let $M = (m_{i,j})$ denote the matrix which represents the transformation $\eta \mapsto f_*\eta$ defined in (4.1), i.e., we set $m_{i,j} = 1$ if the $i$-th basis element in our ordered basis maps to the $j$-th basis element, and 0 otherwise. To compute the characteristic polynomial of $M$, we expand $\det(M - xI)$ by minors down the last column. We obtain

$$\det(M - xI) = -xM_{n+9,n+9} + (-1)^n M_{1,n+9}, \qquad (C.1)$$

where we use the notation $M_{i,j}$ for the $i,j$-minor of the matrix $M - xI$. To evaluate $M_{n+9,n+9}$ and $M_{1,n+9}$, we expand again in minors along the last column to obtain

$$M_{n+9,n+9} = -x \det \hat{m}_1 + (-1)^n \det \hat{m}_2, \qquad M_{1,n+9} = \det \hat{m}_3,$$

where $\hat{m}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2(n) \end{pmatrix}$, $\hat{m}_2 = \begin{pmatrix} B_1 & * \\ 0 & B_2(n) \end{pmatrix}$, and $\hat{m}_3 = \begin{pmatrix} C_1 & * \\ 0 & C_2(n) \end{pmatrix}$. Here $A_1$, $B_1$, and $C_1$ do not depend on $n$, and $A_2(n)$, $B_2(n)$, and $C_2(n)$ are triangular matrices of size $(n-7)\times(n-7)$, $(n-8)\times(n-8)$ and $(n-8)\times(n-8)$ of the form

$$A_2(n) = \begin{pmatrix} -x & 0 & 0 \\ * & \ddots & 0 \\ * & * & -x \end{pmatrix}, \quad B_2(n) = \begin{pmatrix} 1+x & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1+x \end{pmatrix}, \quad C_2(n) = \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\det A_2(n) = (-x)^{n-7}, \quad \det B_2(n) = (1+x)^{n-8}, \quad \text{and } \det C_2(n) = 1. \qquad (C.2)$$

Since $A_1$, $B_1$, and $C_1$ do not depend on $n$, we may compute them using the matrix $M$ from the case $n = 8$ to find

$$\det A_1 = -x^6(x^8 - x^5 - x^3 + 1), \quad \det B_1 = -x^8, \quad \text{and } \det C_1 = x^5 + x^3 - 1. \qquad (C.3)$$

Using (C.2) and (C.3) we find that the characteristic ploynomial of $M$ is equal to

$$(-1)^n \left[x^9(x+1)^{n-8} - x^{n+1}(x^8 - x^5 - x^3 + 1) + x^5 + x^3 - 1\right] = (x^7 - 1)\chi_n(x)/(x^2 - 1),$$

which completes the proof of $(i)$.

For the proof of $(ii)$, we use the ordered basis

$$\{12, 2a3, 34, 45, 1g2, 2e3, 3c4, 04, 01, 03, 14, 25, 061, 072, 081, \ldots, 0(2k-1)2, 0(2k)1\},$$

and for $(iii)$ we use the ordered basis

$$\{23, 3a4, 015, 13, 01, 12, 2d3, 3c4, 02, 04, 162, 073, 081, 192, \ldots, 0(3k-1)1, 1(3k)2\}.$$

Otherwise, the proofs of cases $(ii)$ and $(iii)$ are similar. We omit the details. □

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Indiana University
Bloomington, IN 47405
`bedford@indiana.edu`

Florida State University
Tallahassee, FL 32306
`kim@math.fsu.edu`