1. (Attributed by Lang to John Tate.) Suppose a map \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous and has the following property: There exists a real number \( C > 0 \) such that for all \( x, y \in \mathbb{R}^n \),

\[
\|f(x + y) - f(x) - f(y)\| \leq C.
\]

Prove that there exists a unique linear map \( g : \mathbb{R}^n \to \mathbb{R}^m \) such that \( g - f \) is bounded in the sup norm. (Hint: Show that for all \( x \),

\[
g(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]

exists – to do this, show that the sequence is Cauchy.)

2. Let \( \bar{B}_r \) denote the closed ball of radius \( r \) in \( \mathbb{R}^n \) centered at 0. Suppose \( f : \bar{B}_r \to \mathbb{R}^n \) satisfies:

(a) \( \|f(x) - f(y)\| \leq b \|x - y\| \), with \( 0 < b < 1 \).

(b) \( \|f(0)\| \leq r(1 - b) \).

Show that there exists a unique \( x \in \bar{B}_r \) such that \( f(x) = x \).

3. Assume the situation of the previous exercise. In addition, let \( c \) be a positive real number, and let \( g : \bar{B}_r \to \mathbb{R}^n \) be a map satisfying \( \|g(x) - f(x)\| \leq c \) for all \( x \in \bar{B}_r \). Assume \( g \) has a fixed point \( x_2 \), and let \( x_1 \) be the fixed point of \( f \) (whose existence you proved in the previous exercise). Prove that \( \|x_2 - x_1\| \leq c/(1 - b) \).

4. Define \( f : L(\mathbb{R}^n, \mathbb{R}^n) \to L(L(\mathbb{R}^n, \mathbb{R}^n), L(\mathbb{R}^n, \mathbb{R}^n)) \) by \( f(u) = \tau_u \), where \( \tau_u(v) = u \circ v \). Prove that \( f \) is continuous with respect to the operator norm defined in class.