1. (a) Prove that 0 is a regular value of the function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by
\[
f(x, y) = y^2 - x(x + 1)(x + 2).
\]
Conclude that the curve \( y^2 = x(x + 1)(x + 2) \) in \( \mathbb{R}^2 \) is a smooth 1-dimensional manifold. Sketch this curve.

(b) Is 0 a regular value of the function \( g : \mathbb{R}^2 \to \mathbb{R} \) defined by \( g(x, y) = y^2 - x^2(x + 1) \)? Why or why not? Sketch the curve \( y^2 = x^2(x + 1) \).

NOTE: (a) is an example of a “nonsingular elliptic curve”, while (b) is a “singular elliptic curve”.

2. Let \( f : M \to N \) denote a smooth map between manifolds and suppose \( g : M \to M \) is a diffeomorphism which satisfies \( f \circ g = f \). Given a point \( x \in M \), prove that \( x \) is a regular point of \( f \) if and only if \( g(x) \) is a regular point of \( f \).

For the following problems, define the following terms: \( M_{n \times n}(\mathbb{R}) \) denotes the set of \( n \times n \) matrices with real entries. \( A' \) denotes the transpose of a square matrix \( A \).

3. Define \( O(n, \mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) : A'A = I_n \} \). Prove \( O(n, \mathbb{R}) \) is a smooth manifold, and find its dimension. Hint: Let \( S_n(\mathbb{R}) \) denote the set of \( n \times n \) symmetric matrices. Define \( f : M_{n \times n}(\mathbb{R}) \to S_n(\mathbb{R}) \) by \( f(A) = A'A \). First, prove that \( I \) is a regular point of \( f \). To do this, I would recommend calculating \( df_I(V) \) for an arbitrary matrix \( V \). Then show directly that \( df_I \) is onto. Now let \( A \) be an arbitrary element of \( O(n, \mathbb{R}) \). Show that \( L_A : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R}) \) defined by \( L_A(X) = AX \) is a diffeomorphism. Now use the result of 2.

NOTE: The result of 3. gives a space that is a manifold and a group at the same time (and in which the functions \( (g, h) \mapsto gh \) and \( g \mapsto g^{-1} \) are smooth). Such spaces are called Lie groups and are extremely important in geometry and topology.

4. As many of you know, a covering map is a continuous surjective map \( f : X \to Y \), where \( X \) and \( Y \) are topological spaces, with the following property: For each \( y \in Y \), there exists an open neighborhood \( V \) of \( y \) in \( Y \) such that
\[
f^{-1}(V) = \bigcup_{\alpha \in A} U_\alpha,
\]
where the sets \( U_\alpha \) are pairwise disjoint open sets, and for each \( \alpha \in A \), the restriction \( f : U_\alpha \to V \) is a homeomorphism. Prove that if \( M \) and \( N \) are smooth manifolds of the same dimension, and \( M \) is compact and nonempty, and \( N \) is connected, and \( f : M \to N \) is a smooth map for which every \( x \in M \) is a regular point, then \( f : M \to N \) is a covering map. Give a counterexample showing that you cannot drop the hypothesis that \( M \) is compact.