Modified Pressure-Correction Projection Methods: Open Boundary and Variable Time Stepping

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Abstract. In this paper, we design and study two modifications of the first order standard pressure increment projection scheme for the Stokes system. The first scheme improves the existing schemes in the case of open boundary condition by modifying the pressure increment boundary condition, thereby minimizing the pressure boundary layer and recovering the optimal first order decay. The second scheme allows for variable time stepping. It turns out that the straightforward modification to variable time stepping leads to unstable schemes. The proposed scheme is not only stable but also exhibits the optimal first order decay. Numerical computations illustrating the theoretical estimates are provided for both new schemes.

1 Introduction

We consider the time-dependent Stokes system on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\partial\Omega$ and over a finite time interval $[0, T]$. For a given force $\mathbf{f} : \Omega \times [0, T] \to \mathbb{R}^d$, the velocity $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ and the pressure $p : \Omega \times [0, T] \to \mathbb{R}$ are related via the following system

$$\rho\partial_t \mathbf{u} - 2\mathrm{div}\left(\mu\nabla^S\mathbf{u}\right) + \nabla p = f \quad \text{and} \quad \mathrm{div}(\mathbf{u}) = 0 \qquad \text{in } \Omega \times [0, T], \tag{1}$$

where ρ and μ are the fluid density and viscosity of the fluid assumed to be constant (and positive) and $\nabla^S := \frac{1}{2}\left(\nabla + \nabla^T\right)$ denotes the symmetric part of the gradient. Relations (1) is supplemented by a boundary condition either prescribing the velocity or the force at the boundary. In order to simplify the presentation, we consider homogeneous cases

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega \times [0, T] \tag{2}$$

or

$$(2\mu\nabla^S\mathbf{u} - p)\boldsymbol{\nu} = 0 \qquad \text{on } \partial\Omega \times [0, T], \tag{3}$$

where $\boldsymbol{\nu}$ is the unit, outward pointing normal of $\partial\Omega$. In addition, the initial velocity $\mathbf{u}_0 : \Omega \to \mathbb{R}^d$ is prescribed, i.e. $\mathbf{u}(0, .) := \mathbf{u}_0$. At this point, we note that the extension to the Navier-Stokes system is treated similarly with the additional, but well known, techniques used to cope with the additional nonlinearity.

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Most projection methods are based on the original ideas of Chorin [1] and Temam [9], see also Goda [2]. We refer to [4] for an overview of projection methods.

In this work, we obtain two different results regarding the so-called incremental pressure correction schemes studied for instance in [5–7,3]:

- The scheme proposed in [3] when the system is subject to open boundary conditions, see (3), is suboptimal with respect to the time discretization parameter. We propose and study a new scheme able to recover the optimal convergence rate, see Figure 1.
- We analyze a new scheme allowing for variable time stepping. It turns out that the straightforward generalization of constant time stepping to variable time stepping is unstable, see Figure 2. To the best of our knowledge, projection schemes with variable time stepping have not been studied in the literature. Notice however, that no additional difficulty arises from having variable time stepping in the non-incremental scheme setting.

Given a positive integer N, let $0 = t^0 < t^1 < t^2 < \cdots < t^N = T$ be a subdivision of the time interval $[0, T]$ and set $\delta t^n := t^n - t^{n-1}$. The norm in $L_2(\Omega)$ is denoted by $\|.\|_0$ and we equip $H^1(\Omega)$ with the norm $\|.\|_1 := \left(\|.\|_0^2 + \|\nabla.\|_0^2\right)^{1/2}$. In addition, given a sequence of function $\varphi_{\delta t} := \{\varphi^n\}_{n=0}^N$, we define the following discrete (in time) norms:

$$\|\varphi_{\delta t}\|_{l^2(E)} := \left(\sum_{n=0}^{N} \delta t^n \|\varphi^n\|_E^2\right)^{1/2}, \qquad \|\varphi_{\delta t}\|_{l^\infty(E)} := \max_{0\le n\le N}(\|\varphi^n\|_E). \tag{4}$$

for $E := L_2(\Omega)$ or $H^1(\Omega)$.

2 Optimal Incremental Projection Scheme for Open Boundary problem

We consider the system (1) supplemented with the force condition at the boundary (3) and focus on the case of uniform (constant) time steps, i.e. $\delta t := \frac{T}{N} = \delta t^n$, $n = 0, \cdots, N$. The case of variable time steps is discussed in Section 3. The approximations of $\mathbf{u}(t^n, .)$ and $p(t^n, .)$, $n = 0, ..., N$, are denoted $\mathbf{u}^n$ and p^n respectively. For clarity, we also denote by ϕ^n the pressure increment approximation, i.e.

$$p^n = p^{n-1} + \phi^n. \tag{5}$$

Together with the initial condition on the velocity $\mathbf{u}^0 = \mathbf{u}_0$, the algorithm requires initial pressure $p(0)$ and we set $p^{-1} := p^0 := p(0)$, and so $\phi^0 := 0$. We seek recursively the velocity $\mathbf{u}^{n+1}$ and the pressure p^{n+1} in three steps. First, given $\mathbf{u}^n$, ϕ^n and p^n, the velocity approximation at t^{n+1} is given by

$$\rho\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - 2\mathrm{div}(\mu\nabla^S\mathbf{u}^{n+1}) + \nabla(p^n + \phi^n) - \alpha\nabla\mathrm{div}\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t}\right) = f(t^{n+1}, .), \tag{6}$$

in Ω, where $\alpha \geq 1$ is a stabilization parameter. As we shall see, the consistent "grad-div" term is instrumental to ensure the stability of the scheme by providing a control on $\|\phi^{n+1} - \phi^n\|_{H^1(\Omega)}$, i.e. the second increment of the pressure; see (13).

Equation (6) is supplemented by the boundary condition

$$\left(2\mu\nabla^S\mathbf{u}^{n+1} - (p^n + \phi^n) + \alpha\mathrm{div}\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t}\right)\right)\boldsymbol{\nu} = 0 \qquad \text{on } \partial\Omega. \tag{7}$$

The second step consist in seeking the new pressure increment approximation ϕ^{n+1} as the solution to

$$-\delta t \Delta\phi^{n+1} + \delta t\phi^{n+1} = -\mathrm{div}(\mathbf{u}^{n+1}) \quad \text{in } \Omega \tag{8}$$

together with the boundary condition

$$\frac{\partial}{\partial\nu}\phi^{n+1} = 0 \quad \text{on } \partial\Omega. \tag{9}$$

Finally, the new pressure approximation is then given by (5).

The novelty of this projection scheme is to impose a Neuman boundary condition on the pressure increment (and therefore on the pressure). Its aim is to reduce the boundary layer on the pressure and improve the convergence rate. Compare with [3] where a Dirichlet condition $p^{n+1} = p^n$ is proposed on the pressure. This is at the expense of adding (i) an harmless zero order term $\delta t\phi^{n+1}$ in (8) to be able to recover the full $l^2(H^1(\Omega))$ norm for the pressure and (ii) the more serious "grad-div" stabilization term in (6), which complicates the linear algebra. Notice that the boundary condition (9) proposed here corresponds to the standard boundary condition when the velocity is imposed at the boundary; refer to [3].

We now briefly discuss the stability and error estimates for the scheme (6)-(9).

Theorem 1 (Velocity Stability). *Set* $\mathbf{f} \equiv 0$ *and assume* $\alpha \geq 1$*, then there holds*

$$\rho\|\mathbf{u}_{\delta t}\|^2_{l^\infty(L_2(\Omega))} + 4\mu\|\nabla^S\mathbf{u}_{\delta t}\|^2_{l^2(L_2(\Omega))} + \alpha\|\mathrm{div}(\mathbf{u}_{\delta t})\|^2_{l^\infty(L_2(\Omega))} + (\delta t)^2\|p_{\delta t}\|^2_{l^\infty(H^1(\Omega))}$$
$$\leq \rho\|\mathbf{u}_0\|_0^2 + \alpha\|\mathrm{div}(\mathbf{u}_0)\|_0^2 + (\delta t)^2\|p_0\|_1^2$$

provided $\mathbf{u}_0 \in L_2(\Omega)^d$, $\mathrm{div}(\mathbf{u}_0) \in L_2(\Omega)$ *and* $p_0 \in H^1(\Omega)$.

Proof. Multiplying (6) by $2\delta t\mathbf{u}^{n+1}$ and integrating over Ω one gets after integrating by parts and using the boundary condition (7)

$$\begin{aligned}&\rho\left(\|\mathbf{u}^{n+1}\|_0^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_0^2 - \|\mathbf{u}^n\|_0^2\right) + 4\delta t\mu\|\nabla^S\mathbf{u}^{n+1}\|_0^2\\ &\quad + \alpha\left(\|\mathrm{div}(\mathbf{u}^{n+1})\|_0^2 + \|\mathrm{div}(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_0^2 - \|\mathrm{div}(\mathbf{u}^n)\|_0^2\right)\\ &\quad - 2\delta t\int_\Omega (p^n + \phi^n)\mathrm{div}(\mathbf{u}^{n+1})d\mathbf{x} = 0.\end{aligned} \tag{10}$$

The last term in the left hand side of the above relation is estimated upon multiplying (8) by $2\delta t(p^n + \phi^n)$, integrating over Ω and using the boundary condition (9)

$$\begin{aligned}-2\delta t\int_\Omega (p^n + \phi^n)\mathrm{div}(\mathbf{u}^{n+1})d\mathbf{x} &= 2(\delta t)^2\int_\Omega \nabla\phi^{n+1}\cdot\nabla(p^n + \phi^n)d\mathbf{x}\\ &\quad + 2(\delta t)^2\int_\Omega \phi^{n+1}(p^n + \phi^n)d\mathbf{x}.\end{aligned}$$

In view of (5), we write $p^n + \phi^n = \phi^n - \phi^{n+1} + p^{n+1}$ and realize that

$$-2\delta t \int_\Omega (p^n + \phi^n)\mathrm{div}(\mathbf{u}^{n+1})d\mathbf{x} = (\delta t)^2\|\phi^n\|_1^2 - (\delta t)^2\|\phi^{n+1} - \phi^n\|_1^2 + (\delta t)^2\|p^{n+1}\|_1^2 - (\delta t)^2\|p^n\|_1^2. \tag{11}$$

It remains to derive a bound for $\|\phi^{n+1} - \phi^n\|_1$. Multiplying by $\phi^{n+1} - \phi^n$ the difference of two successive relations (8) and integrating over Ω yield

$$\delta t\|\phi^{n+1} - \phi^n\|_1^2 = -\int_\Omega \mathrm{div}(\mathbf{u}^{n+1} - \mathbf{u}^n)(\phi^{n+1} - \phi^n)d\mathbf{x}, \tag{12}$$

after an integration by parts and taking advantage of the boundary condition (9). Hence, we deduce that

$$\delta t\|\phi^{n+1} - \phi^n\|_1 \leq \|\mathrm{div}(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_0. \tag{13}$$

Gathering the estimate (13), (11) and (10), we obtain

$$\begin{aligned}\rho&\left(\|\mathbf{u}^{n+1}\|_0^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_0^2 - \|\mathbf{u}^n\|_0^2\right) + 4\delta t\mu\|\nabla^S\mathbf{u}^{n+1}\|_0^2 \\ &+ \alpha\left(\|\mathrm{div}(\mathbf{u}^{n+1})\|_0^2 - \|\mathrm{div}(\mathbf{u}^n)\|_0^2\right) + (\alpha-1)\|\mathrm{div}(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_0^2 \\ &+ (\delta t)^2\left(\|p^{n+1}\|_1^2 - \|p^n\|_1^2 + \|\phi^n\|_1^2\right) \leq 0.\end{aligned}$$

The desired bound follows after summing for $n = 0$ to $N-1$. □

We emphasize that the above proof is closely related to the case where Dirichlet boundary conditions are imposed on the velocity; refer for instance to [4,8]. The difference resides on the fact that (13) can be circumvented using an integration by parts in (12). Hence following the techniques developed for the Dirichlet case together with the argumentation leading to (13) yields the optimal convergence rates

$$\begin{aligned}\max_{n=1,\ldots,N}&\|\mathbf{u}(t^n,.) - \mathbf{u}^n\|_{L^2(\Omega)} + \left(\sum_{n=1}^N \delta t\|\nabla^S(\mathbf{u}(t^n,.) - \mathbf{u}^n)\|_{L_2(\Omega)}^2\right)^{1/2} \\ &+ \alpha\max_{n=1,..,N}\|\mathrm{div}(\mathbf{u}(t^n,.) - \mathbf{u}^n)\|_{L_2(\Omega)} + \left(\sum_{n=1}^N \delta t\|p(t^n) - p^n\|_{L_2(\Omega)}^2\right)^{1/2} \leq C\delta t,\end{aligned}$$

with a constant C independent of N and provided the exact velocity $\mathbf{u}$ and pressure p satisfy the appropriate regularity conditions.

To illustrate the optimality of the proposed algorithm, we consider the exact solution

$$\mathbf{u}(t,x,y) := \begin{pmatrix}\sin(t+x)\sin(t+y) \\ \cos(t+x)\cos(t+y)\end{pmatrix}, \quad p(t,x,y) = \sin(t+x-y)$$

defined $\Omega := (0,1)^2$. The behavior of the errors in velocity and pressure approximations versus the time step δt used are depicted in Figure 1. Suboptimal order of convergence $\mathcal{O}(\delta t^{1/2})$ is observed for the standard method while the optimal order of convergence $\mathcal{O}(\delta t)$ is recovered using the proposed scheme. The space discretization is chosen fine enough not to interfere with the time discretization error.

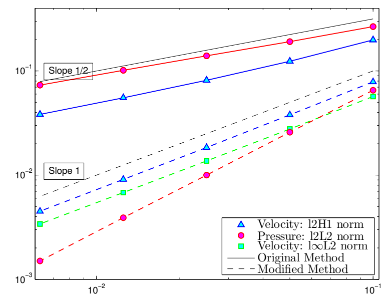

Fig. 1. Decay of different error norms versus δt for the original and modified standard pressure correction projection method. Suboptimal order of convergence $\mathcal{O}(\delta t^{1/2})$ is observed for the standard method while the optimal order of convergence $\mathcal{O}(\delta t)$ is recovered using the proposed scheme.

3 Variable Time Stepping

We now consider variable time steps δt^n satisfying

$$\delta t^n \leq \overline{\delta t}, \qquad 1 \leq n \leq N,$$

for a positive constant $\overline{\delta t}$ independent of n. The incremental projection scheme with variable time stepping reads as follow. Given $\mathbf{u}^n$, ϕ^n and p^n, the velocity approximation at t^{n+1} is defined by the relation

$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t^{n+1}} - 2\mathrm{div}(\mu \nabla^S \mathbf{u}^{n+1}) + \nabla (p^n + \frac{(\overline{\delta t})^2}{\delta t^n \delta t^{n+1}} \phi^n) = f(t^{n+1}, .). \quad \text{in } \Omega \quad (14)$$

For simplicity, we consider the boundary condition $\mathbf{u} = 0$ on $\partial\Omega$ but the techniques presented in Section 2 for the open boundary condition case apply in this context as well. The pressure increment ϕ^{n+1} solves

$$-\frac{(\overline{\delta t})^2}{\delta t^{n+1}} \Delta \phi^{n+1} = -\rho \mathrm{div}(\mathbf{u}^{n+1}) \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial}{\partial \nu} \phi^{n+1} = 0 \quad \text{on } \partial\Omega. \quad (15)$$

Finally, the pressure is updated according to relation (5).

The standard pressure correction schemes are derived from the original velocity prediction - projection scheme, see for instance [4]. When the same time step value is used for the velocity prediction and correction, the factors multiplying the increment ϕ^n in (14) and (15) becomes $\frac{\delta t^n}{\delta t^{n+1}}$ and δt^{n+1} instead of $\frac{\overline{\delta t}^2}{\delta t^n \delta t^{n+1}}$ and $\frac{\overline{\delta t}^2}{\delta t^{n+1}}$ as in the proposed scheme (14)-(15). This alternative is referred as the standard scheme but we emphasize that there is no reason for the projection step to use the velocity prediction time step as projection parameter. In fact, this choice turns out to be numerically unstable as illustrated now. We consider the same setting as in Section 2 but with variable time steps given by

$$\delta t^n = \delta t^1 \times \begin{cases} 1 & \text{when } n \text{ is odd,} \\ 10^{-2} & \text{when } n \text{ is even,} \end{cases} \quad (16)$$

for different values of δt^1. In this case, we set $\overline{\delta t} := \delta t^1$. Figure 2 (left) illustrates the unstable behavior of $\|\mathbf{u}^n\|_{L_2(\Omega)}$ for $n = 0, ..., N$ when using the standard scheme with $\delta t^1 = 0.025$. However, the $l^2(H^1(\Omega))$ and $l^\infty(L_2(\Omega))$ errors on the velocity decay like $\overline{\delta t}$ when the proposed scheme (14) - (15) is used, see Figure 2 (right).

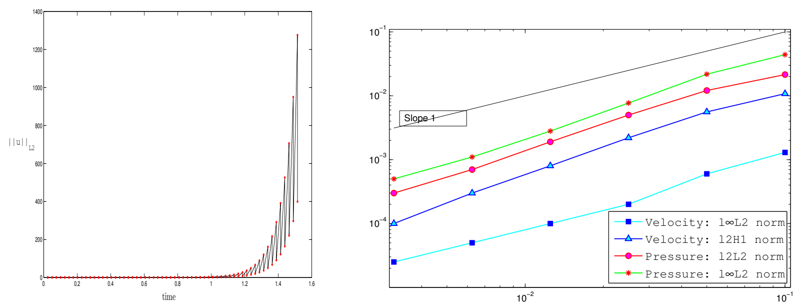

Fig. 2. (Left) Evolution of $\|\mathbf{u}(t^n, .)\|_{L_2(\Omega)}$ when using the standard scheme with $\delta t^1 = 0.025$ and δt^n given by (16). (Right) Decay of the velocity and pressure errors versus $\overline{\delta t}$ and with the time steps δt^n given by (16) when using the proposed scheme. The optimal order of convergence $\mathcal{O}(\overline{\delta t})$ is observed.

We now briefly discuss the stability and error estimates for the scheme (14)-(15).

Theorem 2 (Velocity Stability). *Set $\mathbf{f} \equiv 0$, and assume $\delta t^n \leq \overline{\delta t}$, $n = 1, .., N$, then there holds*

$$\rho\|\mathbf{u}_{\delta t}\|^2_{l^\infty(L_2(\Omega))} + 4\mu\|\nabla^S \mathbf{u}_{\delta t}\|^2_{l^2(L_2(\Omega))} + \frac{1}{\rho}(\overline{\delta t})^2\|p_{\delta t}\|^2_{l^\infty(H^1(\Omega))} \leq \rho\|\mathbf{u}_0\|^2_0 + (\overline{\delta t})^2\|p_0\|^2_1$$

provided $\mathbf{u}_0 \in L_2(\Omega)^d$ and $p_0 \in H^1(\Omega)$.

Proof. Multiplying (14) by $2\delta t^{n+1}\mathbf{u}^{n+1}$ and integrating over Ω one gets after integrating by parts and using the boundary condition $\mathbf{u} = 0$,

$$\begin{aligned} &\rho\left(\|\mathbf{u}^{n+1}\|^2_0 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_0 - \|\mathbf{u}^n\|^2_0\right) + 4\delta t^{n+1}\mu\|\nabla^S \mathbf{u}^{n+1}\|^2_0 \\ &\quad - 2\int_\Omega \left(\delta t^{n+1} p^n + \frac{(\overline{\delta t})^2}{\delta t^n \delta t^{n+1}}\phi^n\right) \mathrm{div}(\mathbf{u}^{n+1}) d\mathbf{x} = 0. \end{aligned} \tag{17}$$

The pressure increment relation (15) is invoked to derive a bound for the last term in the left hand side of the above relation. More precisely, multiplying (15) by $2(\delta t^{n+1}p^n + \frac{(\overline{\delta t})^2}{\delta t^n \delta t^{n+1}}\phi^n)$, integrating over Ω and using the boundary condition (9) we realize that

$$\begin{aligned} &-2\rho\int_\Omega (\delta t^{n+1}p^n + \frac{(\overline{\delta t})^2}{\delta t^n \delta t^{n+1}}\phi^n)\mathrm{div}(\mathbf{u}^{n+1})d\mathbf{x} \\ &\quad = 2(\overline{\delta t})^2\int_\Omega \nabla\phi^{n+1}\cdot\nabla p^n d\mathbf{x} + 2\int_\Omega \nabla\left(\frac{(\overline{\delta t})^2}{\delta t^{n+1}}\phi^{n+1}\right)\cdot\nabla\left(\frac{(\overline{\delta t})^2}{\delta t^n}\phi^n\right) d\mathbf{x}. \end{aligned}$$

Relation (5) allows us to rewrite the right hand side of the above expression as

$$(\overline{\delta t})^2 \left(\|\nabla p^{n+1}\|_0^2 - \|\nabla p^n\|_0^2 - \|\nabla \phi^{n+1}\|_0^2\right) + \frac{(\overline{\delta t})^4}{(\delta t^{n+1})^2}\|\nabla \phi^{n+1}\|_0^2 + \frac{(\overline{\delta t})^4}{(\delta t^n)^2}\|\nabla \phi^n\|_0^2 - \left\|\nabla\left(\frac{(\overline{\delta t})^2}{\delta t^{n+1}}\phi^{n+1} - \frac{(\overline{\delta t})^2}{\delta t^n}\phi^n\right)\right\|_0^2.$$

Going back to (17), we get

$$\begin{aligned}&\rho\left(\|\mathbf{u}^{n+1}\|_0^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_0^2 - \|\mathbf{u}^n\|_0^2\right) + 4\delta t^{n+1}\mu\|\nabla^S \mathbf{u}^{n+1}\|_0^2 \\ &\quad + \frac{1}{\rho}(\overline{\delta t})^2\left(\|\nabla p^{n+1}\|_0^2 - \|\nabla p^n\|_0^2\right) + \frac{1}{\rho}(\overline{\delta t})^2\left(\frac{(\overline{\delta t})^2}{(\delta t^{n+1})^2} - 1\right)\|\nabla \phi^{n+1}\|_0^2 \\ &\quad + \frac{1}{\rho}\frac{(\overline{\delta t})^4}{(\delta t^n)^2}\|\nabla \phi^n\|_0^2 = \frac{1}{\rho}\left\|\nabla\left(\frac{(\overline{\delta t})^2}{\delta t^{n+1}}\phi^{n+1} - \frac{(\overline{\delta t})^2}{\delta t^n}\phi^n\right)\right\|_0^2.\end{aligned}$$

The difference of two successive relations (15) together with the boundary condition $\mathbf{u}^n = \mathbf{u}^{n+1} = 0$ on $\partial\Omega$ guarantee that

$$\left\|\nabla\left(\frac{(\overline{\delta t})^2}{\delta t^{n+1}}\phi^{n+1} - \frac{(\overline{\delta t})^2}{\delta t^n}\phi^n\right)\right\|_0 \leq \rho\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_0.$$

Hence, using the assumption $\delta t^{n+1} \leq \overline{\delta t}$,

$$\begin{aligned}&\rho\left(\|\mathbf{u}^{n+1}\|_0^2 - \|\mathbf{u}^n\|_0^2\right) + 4\delta t^{n+1}\mu\|\nabla^S \mathbf{u}^{n+1}\|_0^2 + \frac{1}{\rho}(\overline{\delta t})^2\left(\|\nabla p^{n+1}\|_0^2 - \|\nabla p^n\|_0^2\right) \\ &\quad + \frac{1}{\rho}\frac{(\overline{\delta t})^4}{(\delta t^n)^2}\|\nabla \phi^n\|_0^2 \leq 0,\end{aligned}$$

and the desired bound follows after summing for $n = 0$ to $N - 1$. □

Regarding the error decay we have that under the assumption $\delta t^n \leq \overline{\delta t}$, $n = 1, .., N$, there exists a constant C independent of n and $\overline{\delta t}$ such that

$$\max_{n=1,\ldots,N}\|\mathbf{u}(t^n,.) - \mathbf{u}^n\|_{L^2(\Omega)} + \left(\sum_{n=1}^N \delta t^n\|\mathbf{u}(t^n,.) - \mathbf{u}^n\|_{H^1(\Omega)}^2\right)^{1/2} \leq C\overline{\delta t},$$

provided $\mathbf{u}$ and p are smooth enough and $\overline{\delta t}$ is sufficiently small. The proof of the above claim is omitted but relies on the argumentations provided in the proof of Theorem 2. In addition, we emphasize that scheme (14) - (15) does not optimize the choice of δt^n in order to equi-distribute the time discretization errors and explain that the decay rate is dictated by $\overline{\delta t}$ (and not δt^n, $n = 1, ..., N$). Including such mechanism is out of the scope of this work. Moreover, the decay rate for the $l^2(L_2(\Omega))$ error on the pressure is still an open problem but the numerical results provided in Figure 2 indicate an optimal rate.

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