Chapter 12. Vectors and the Geometry of Space

12.4 The Cross Product

* Given two nonzero vectors $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$, can we find a nonzero vector $\mathbf{c}$ that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$?

Definition 1 If $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$, then the _____________ of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} :=< a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

Remark 2 Recall the definition of a determinant.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Example 1. If $\mathbf{a} =< 1, 3, 4 >$ and $\mathbf{b} =< 2, 7, -5 >$, then $\mathbf{a} \times \mathbf{b} =$

* Cross product of two vectors is a vector.

Theorem 3 The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Remark 4 The direction of the vector $\mathbf{a} \times \mathbf{b}$: right-hand rule.

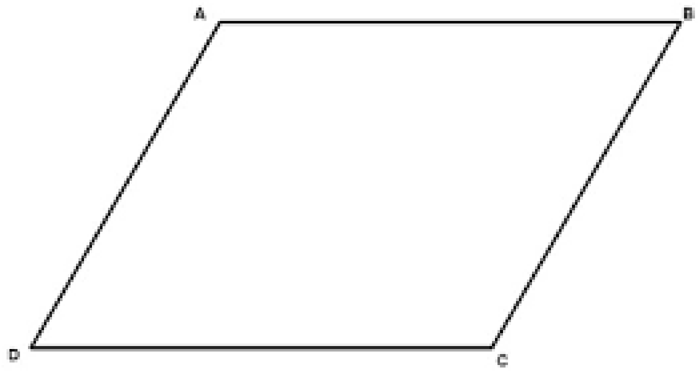
Theorem 5 If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ $(0 \le \theta \le \pi)$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta, \qquad \text{(length)}$$

Corollary 6 The two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

Remark 7 Figure 2. The geometric interpretation



* The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Example 3. Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Example 4. Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Remark:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

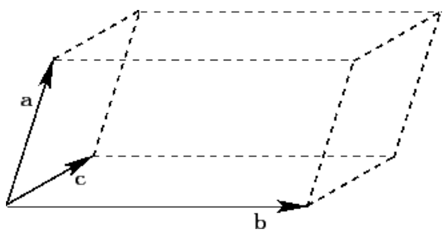
Properties If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (NOTE: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$)
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

12.4.1 Triple Products

Definition 8 Scalar triple product :

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



* The volume of the parallelepiped is the magnitude of their scalar triple product:

$$V = Ah =$$

Definition 9 If the volume of the parallelepiped determined by $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is 0, then the vectors must lie in the same plane; that is, they are ___________.

Example 5. Show that the vectors $\mathbf{a} =< 1, 4, -7 >$ $\mathbf{b} =< 2, -1, 4 >$, and $\mathbf{c} =< 0, -9, 18 >$ are coplanar.

12.4.2 Torque

Definition 10 The torque τ is defined to be the cross product of the position and force vectors

$$\tau = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin.

Remark 11 The magnitude of the torque vector is

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

where θ is the angle between the position and force vectors.

Example 6 A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

Suggested Home Work Problems: 12.4 Exercises (8th edition)

$$\# : 3, 5, 17, 19, 27, 29, 33, 35, 39$$