1 Hall Effect

The classical Hall effect was first observed in the XIX century [1]. A thin metal sample is immersed in a constant uniform strong orthogonal magnetic field, and a constant current $j$ flows through the sample, say, in the $x$-direction. By Flemming’s rule, an electric field is created in the $y$-direction, as the flow of charge carriers in the metal is subject to a Lorentz force perpendicular to the current and the magnetic field. This is called the Hall current.

The equation for the equilibrium of forces in the sample

$$NeE + j \wedge B = 0,$$

defines a linear relation. The ratio of the intensity of the Hall current to the intensity of the electric field is the Hall conductance,

$$\sigma_H = \frac{Ne}{B}.$$  

(1)

In the stationary state, $\sigma_H$ is proportional to the dimensionless filling factor $\nu = \frac{\rho h}{eH}$, where $\rho$ is the 2-dimensional density of charge carriers, $h$ is the Planck constant, and $e$ is the electron charge. More precisely, we have

$$\sigma_H = \frac{\nu}{R_H},$$

(2)

where $R_H = \frac{h}{e^2}$ denotes the Hall resistance, which is a universal constant. This measures the fraction of Landau level filled by conducting electrons in the sample.

1.1 Integer Quantum Hall Effect

In 1980, von Klitzing’s experiment showed that, lowering the temperature below 1 K, quantum effects dominate, and the relation of Hall conductance to filling factor shows plateaux at inte-
ger values, [2]. He was awarded the Nobel Prize in 1985 for this discovery, one of whose applications is the precise measurement of the fine structure constant, $\alpha = \frac{\mu_0 c e^2}{2\hbar}$. Under the above conditions, one can effectively ignore the Coulomb interaction between the electrons and one is reduced to the single electron theory.

Laughlin first suggested that IQHE should have a geometric explanation [3]. One of the early successes of Connes’ theory of Non Commutative Geometry [4] was, in fact, a rigorous mathematical model of the Integer Quantum Hall Effect, which accounts for integer quantization, localization, insensitivity to the presence of disorder, and vanishing of direct conductance at plateaux levels [5]. The quantization of the Hall conductance is indeed geometric in nature and closely related to a well known topological phenomenon of quantization in the geometry of compact 2-dimensional manifolds, the Gauss–Bonnet theorem. This shows that the integral of the curvature is always an integer multiple of a basic quantity: a property that is stable under deformations. In the same spirit, the values of the Hall conductance are obtained as the evaluation of a certain characteristic class, that is, as an index theorem.

1.2 Fractional Quantum Hall Effect

The fractional QHE was discovered by Stormer and Tsui in 1982. The setup is as in the quantum Hall effect, with a strong magnetic field and a very pure sample, which we can think of as an infinite 2-dimensional surface. Then the experiment shows that the same graph of $\frac{1}{2\nu} \sigma_H$ against the filling factor $\nu$ gets “quantized” at certain fractional values. Together with Laughlin, they were awarded the Nobel Prize in 1998 for their discovery. Under the above conditions, one either has to incorporate the Coulomb interaction between the electrons and study a many-electron theory, or one has to incorporate an effective interaction term into the single electron model. It is this approach that we will adopt.

We work in a model with non–interacting particles, along the lines of the model that Bellissard and collaborators developed for the IQHE [5], but we will change the underlying geometry to account for the presence of interaction.

1.3 Euclidean plane model

A Euclidean plane model was used by Bellissard and coauthors to give a satisfactory explanation of the IQHE. In this model, noncommutative geometry arises naturally when one considers the Hamiltonian $H$ for a single electron subject to the magnetic field, with an additional potential representing the lattice of ions in the conductor. In a perfect crystal and in the absence of magnetic field, there is a group of translational symmetries, that is, a group of unitary operators $U(a)$, for $a \in G$, a locally compact group of symmetries. Turning on the magnetic field breaks this symmetry, in the sense that translates of the Hamiltonian $H_a = U(a)HU(a)^*$ no longer commute with $H$. The algebra of observables must then include all the resolvents $R_a(z) = U(a)(zI - H)^{-1}U(a)^*$. This implies that, by effect of the magnetic field, the Brillouin zone becomes a non-commutative space [5], [4], [6].

In a discrete model, the Hamiltonian is given in terms of a special Harper operator associated to a lattice in the Euclidean plane. Consider an embedding of $\Gamma = \mathbb{Z}^2$ in the plane as a square lattice.

The Harper operator on the square lattice is
1.4 Random Walk and Harper operators

Let $\Gamma$ be a finitely generated discrete group and $G$ be its Cayley graph, i.e. the vertices of $G$ are the elements of $\Gamma$ and the edges emanating from a vertex $\alpha \in G$ are translates of the vertex $g_i\alpha$ by a generating set, where $\{g_i\}_{i=1}^N$ is a symmetric set of generators of $\Gamma$.

Let $\sigma$ be a multiplier on $\Gamma$, that is,

$$\sigma : \Gamma \times \Gamma \to U(1)$$

is a $U(1)$-valued 2-cocycle on the group $\Gamma$. This means that $\sigma$ satisfies the following identities:

- $\sigma(\gamma, 1) = \sigma(1, \gamma) = 1 \quad \forall \gamma \in \Gamma$
- $\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1 \gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2 \gamma_3)\sigma(\gamma_2, \gamma_3)$
  \quad $\forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma$.

Consider the Hilbert space of square summable functions on $\Gamma$,

$$\ell^2(\Gamma) = \left\{ f : \Gamma \to \mathbb{C}, \sum_{\gamma \in \Gamma} |f(\gamma)|^2 < \infty \right\}.$$ 

There are natural left $\sigma$-regular and right $\sigma$-regular representations on $\ell^2(\Gamma)$.

The left $\sigma$-regular representation is obtained by setting, $\forall \gamma, \gamma' \in \Gamma$,

$$(L_\sigma^\gamma f)(\gamma') = f(\gamma^{-1} \gamma') \sigma(\gamma, \gamma^{-1} \gamma')$$

$$L_\sigma^\gamma L_\sigma^{\gamma'} = \sigma(\gamma, \gamma') L_\sigma^{\gamma \gamma'}.$$ 

Similarly, the right $\sigma$-regular representation is obtained by setting, $\forall \gamma, \gamma' \in \Gamma$,

$$(R_\sigma^\gamma f)(\gamma') = f(\gamma' \gamma) \sigma(\gamma, \gamma')$$

$$R_\sigma^\gamma R_\sigma^{\gamma'} = \sigma(\gamma, \gamma') R_\sigma^{\gamma \gamma'}.$$ 

When $\sigma = 1$, these are the standard left and right regular representations. The cocycle
identity can be used to show that the left $\sigma$-regular representation commutes with the right $\sigma$-regular representation, where $\sigma$ denotes the conjugate cocycle. Also the left $\bar{\sigma}$-regular representation commutes with the right $\sigma$-regular representation.

For $\{g_1, \ldots, g_N\}$ a symmetric set of generators for $\Gamma$ as above, the Random Walk operator on the Cayley graph of $\Gamma$ is the average of the values of the functions evaluated at the nearest neighbors, that is, $H : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is a bounded operator of the form

$$Hf(\gamma) = \sum_{i=1}^{N} f(\gamma g_i) \quad \text{i.e.} \quad H = \sum_{i=1}^{N} R_{g_i},$$

(7)

where $R_{g_i}$ denotes translation by $g_i$ in the right regular representation. The operator $N - H$ is the discrete analog of the Laplacian.

The Harper operator can be viewed as a generalization of the Random Walk operator. It is the Random Walk operator in the $\sigma$-regular representation, i.e.

$$H_{\sigma} = \sum_{i=1}^{N} R_{g_i}^\sigma.$$  

(8)

The operator $N - H$ is the discrete analog of the magnetic Laplacian (cf. [7]).

In the case of the group $\mathbb{Z}^2$, with Cayley graph the square lattice, the multiplier is

$$\sigma((m', n'), (m, n)) = \exp(-i(\alpha_1 m'n + \alpha_2 n'm)),$$

(9)

and the Harper operator $H_{\sigma}$ of (4) is of the form (8), since the operators $U$ and $V$ of (5) are in fact given by $U \equiv R_{(0,1)}^\sigma$ and $V = R_{(1,0)}^\sigma$.

2 Hyperbolic plane model

A model of the quantum Hall effect based on hyperbolic instead of Euclidean geometry was introduced in [8] in the torsion free case. It was then extended in [10] [11] for the more general case with torsion, to give a noncommutative geometry model of the fractional QHE. This model also naturally includes the model of Bellissard et al. for the IQHE.

The hyperbolic metric is introduced to model geometrically the homogeneous effective interaction of all the charge carriers on the given electron (which is not uncommon an assumption in solid state physics). In other words, by effect of electron correlation, a single electron ‘sees’ the surrounding geometry as hyperbolic.

Let $\mathbb{H}$ denote the hyperbolic plane (2D). Its geometry is described as follows. Consider the pseudosphere $\{x^2 + y^2 + z^2 - t^2 = 1\}$ in 4-dimensional Minkowski space-time $M$. The $z = 0$ slice of the pseudosphere realizes an isometric embedding of the hyperbolic plane $\mathbb{H}$ in $M$. In this geometry, a periodic lattice on the resulting surface is determined by a Fuchsian group $\Gamma$ of isometries of $\mathbb{H}$ of signature $(g; \nu_1, \ldots, \nu_n)$. This is a discrete subgroup $\Gamma \subset \text{PSL}(2, \mathbb{R})$ with

Figure 3: The Harper operator on a lattice in the hyperbolic plane.
generators $a_i, b_i, c_j$ and relations

$$\prod_{j=1}^{g} [a_i, b_i] c_1 \cdots c_n = 1 \quad \text{and} \quad c_j^3 = 1.$$  

The quotient $\Sigma(g; \nu_1, \ldots, \nu_n) := \Gamma \backslash \mathbb{H}$ is a hyperbolic orbifold, namely a compact Riemann surface of genus $g$ with $n$ cone points $\{x_1, \ldots, x_n\}$, which are the images of points in $\mathbb{H}$ with non-trivial stabilizer of the action of $\Gamma$. In the torsion free case, where we only have generators $a_i$ and $b_i$, we obtain smooth compact Riemann surfaces of genus $g$.

The magnetic field $B$ gives a $\Gamma$-invariant exact 2-form $B = dA$ on $\mathbb{H}$. The potential $A$ is not $\Gamma$-invariant. However, since we have $\gamma^* B - B = d(\gamma^* A - A) = 0$, we know that $\gamma^* A - A$ is a closed 1-form on the simply connected manifold $\mathbb{H}$. Thus, we have $\gamma^* A - A = d\psi_\gamma$, for all $\gamma \in \Gamma$, with $\psi_\gamma$ a smooth function on $\mathbb{H}$. This function is defined up to a constant, so we can choose the normalization condition that, for a fixed $x_0 \in \mathbb{H}$, $\psi_\gamma(x_0) = 0$ for all $\gamma \in \Gamma$. It follows that $\psi_\gamma$ is real-valued and that $\psi_e(x) \equiv 0$, where $e$ denotes the identity element of $\Gamma$. It is also easy to check that

$$\psi_\gamma(x) + \psi_{\gamma'}(\gamma x) - \psi_{\gamma' \gamma}(x)$$

is independent of $x \in \mathbb{H}$, $\forall \gamma, \gamma' \in \Gamma$. Thus, $\psi$ defines a multiplier on $\Gamma$ by setting

$$\sigma(\gamma, \gamma') = \exp(i\psi_\gamma(\gamma' x_0)).$$

### 2.1 A discrete model of FQHE

The discrete analogue of the Schrödinger equation describing the quantum mechanics of a single electron confined to move along the Cayley graph of $\Gamma$ (embedded in $\mathbb{H}$) subject to the periodic magnetic field $B$ is

$$i \frac{\partial}{\partial t} \psi = H_\sigma \psi + V \psi, \quad (10)$$

where all physical constants have been set equal to 1. Here $H_\sigma$ is the Harper operator encoding the magnetic field and $V$ is the electric potential. The continuous model using the magnetic Schrödinger operator gives equivalent results, cf. [8].

### 2.2 Algebra of observables

The electric potential $V$ in the equation (10) is taken to be an operator in the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$, which consists of functions

$$f : \Gamma \to \mathbb{C}$$

with product

$$f_1 * f_2(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} f_1(\gamma_1)f_2(\gamma_2)\sigma(\gamma_1, \gamma_2),$$

acting on the Hilbert space $\ell^2(\Gamma)$.

The algebra of observables is the $C^*$ algebra generated by all the Hamiltonians

$$H_{\sigma, V} = H_\sigma + V, \quad \forall V \in \mathbb{C}(\Gamma, \sigma), \quad V = V^*.$$  

This is the twisted group $C^*$-algebra $C^*_r(\Gamma, \sigma)$ described below.

To understand the structure of this algebra, we recall some facts about some classical von Neumann and $C^*$-algebras.

Let $\mathcal{U}(\Gamma, \sigma)$ be defined as

$$\mathcal{U}(\Gamma, \sigma) = \{ A \in B(\ell^2(\Gamma)) : [L_\gamma^\sigma, A] = 0 \forall \gamma \in \Gamma \},$$

that is, $\mathcal{U}(\Gamma, \sigma)$ is the commutant of the left $\sigma$-regular representation. By general theory, it is a von Neumann algebra, and it is called the **twisted group von Neumann algebra**. It can also be realized in the following manner: the right $\sigma$-regular representation of $\Gamma$ extends to a $*$ representation of the twisted group algebra, $\mathbb{C}(\Gamma, \sigma) \to$
follows that the weak closure (which coincides with the strong closure) of \( \mathbb{C}(\Gamma, \sigma) \) also yields the twisted group von Neumann algebra \( \mathcal{U}(\Gamma, \sigma) \), by the commutant theorem of von Neumann.

The norm closure of \( \mathbb{C}(\Gamma, \sigma) \) yields the (reduced) twisted group \( C^* \) algebra \( C^*_r(\Gamma, \sigma) \).

In the simpler case of the Euclidean plane model of the IQHE, where \( \Gamma = \mathbb{Z}^2 \), for \( \sigma = 1 \) we have \( \mathcal{U}(\Gamma, 1) \cong L^\infty(\mathbb{T}^2) \) and \( C^*(\Gamma, 1) \cong C(\mathbb{T}^2) \), while, when we turn on the magnetic field, we obtain \( \sigma \) as in (9) and the twisted group \( C^* \) algebra can be identified with the non commutative torus, that is, the irrational rotation algebra

\[
C^*_r(\mathbb{Z}^2, \sigma) = A_{\theta} ,
\]

with \( \theta = \alpha_2 - \alpha_1 \), as before.

The key properties of the algebras \( C^*_r(\Gamma, \sigma) \) and \( \mathcal{U}(\Gamma, \sigma) \) are summarized as follows:

- \( \mathcal{U}(\Gamma, \sigma) \) is generated by its projections and it is also closed under the measurable functional calculus i.e. if \( A \in \mathcal{U}(\Gamma, \sigma) \) and \( A = A^* \), \( A > 0 \), then \( f(A) \in \mathcal{U}(\Gamma, \sigma) \) for all essentially bounded measurable functions \( f \) on \( \mathbb{R} \).

- On the other hand, \( C^*_r(\Gamma, \sigma) \) has only at most countably many projections and is only closed under the continuous functional calculus.

### 2.3 Spectral theory

A fundamental part of the theory of QHE is understanding the gaps in the energy spectrum.

For a Harper operator, since the set of generators \( \{g_i\}_{i=1}^N \) of the group \( \Gamma \) is symmetric, it follows that \( H_\sigma \) is a bounded self-adjoint operator on \( \ell^2(\Gamma) \). Therefore its spectrum \( \text{spec}(H_\sigma) \) is a closed and bounded subset of \( \mathbb{R} \). It follows that the complement \( \mathbb{R} \setminus \text{spec}(H_\sigma) \) is an open subset of \( \mathbb{R} \), so in particular, it is the countable union of disjoint open intervals. Each such interval is called a gap in the spectrum of \( H_\sigma \).

A fundamental question is how many gaps there are in the spectrum of \( H_{\sigma, V} \). Remarkably, the answer to this question depends on whether or not the multiplier \( \sigma \) determines a rational cohomology class, that is, on the condition \( [\sigma] \in H^2(\Gamma, \mathbb{Q}/\mathbb{Z}) \). In 2D, this condition is the same as requiring that the magnetic flux \( \theta = \langle [\sigma], [\Gamma] \rangle \) is rational! (Here \([\Gamma]\) denotes the fundamental class in the homology of \( \Gamma \).)

In our model \( \Gamma \) is a cocompact Fuchsian group of signature \((g, \nu_1, \ldots, \nu_n)\). In this case, we proved in [10] that, if \( [\sigma] \) is rational, then there is only a finite number of gaps in the spectrum of \( H_\sigma + V \). In fact, if \( \theta = p/q \) then the number of gaps is at most

\[
(q + 1) \prod_{j=1}^{n} (\nu_j + 1) .
\]

In terms of the algebra of observables, the question of how many gaps there are in the spectrum of \( H_{\sigma, V} \) can be reduced to studying the number of projections in the \( C^* \)-algebra \( C^*_r(\Gamma, \sigma) \) (up to equivalence).

In fact, we have

\[
H_{\sigma, V} \in \mathbb{C}(\Gamma, \sigma) \subset C^*_r(\Gamma, \sigma) \subset \mathcal{U}(\Gamma, \sigma) .
\]

In particular, \( H_\sigma \) and its spectral projections

\[
P_E = \chi_{(-\infty, E]}(H_{\sigma, V})
\]

belong to the algebra \( \mathcal{U}(\Gamma, \sigma) \). Moreover, when \( E \not\in \text{spec}(H_{\sigma, V}) \), the spectral projection \( P_E \) is in \( C^*_r(\Gamma, \sigma) \). In fact, suppose that the spectrum of \( H_{\sigma, V} \) is contained in a closed interval, and that the open interval \((a, b)\) is a spectral gap of \( H_{\sigma, V} \).
Suppose that \( E \in (a,b) \), i.e. \( E \notin \text{spec}(H_{\sigma,V}) \). Then there is a holomorphic function \( \phi \) on a neighbourhood of \( \text{spec}(H_{\sigma,V}) \) such that

\[
P_E = \phi(H_{\sigma,V}) = \int_C \frac{d\lambda}{\lambda - H_{\sigma,V}} \tag{12}
\]

where \( C \) is a closed contour enclosing the spectrum of \( H_{\sigma} \) to the left of \( E \). Since \( C^*_r(\Gamma,\sigma) \) is closed under the holomorphic functional calculus, it follows that \( P_E \in C^*_r(\Gamma,\sigma) \).

The equivalence relation we need to consider on projections, so that the counting will provide the counting of spectral gaps, is described as follows. Let \( \text{Proj}(C^*_r(\Gamma,\sigma) \otimes \mathcal{K}) \) denote the projections in \( C^*_r(\Gamma,\sigma) \otimes \mathcal{K} \), where \( \mathcal{K} \) the \( C^* \)-algebra of compact operators. Two projections \( P,Q \in \text{Proj}(C^*_r(\Gamma,\sigma) \otimes \mathcal{K}) \) are said to be Murray-von Neumann equivalent if there is an element \( V \in C^*_r(\Gamma,\sigma) \otimes \mathcal{K} \) such that \( P = V^*V \) and \( Q = VV^* \), and we write \( P \sim Q \). It can be shown that \( \text{Proj}(C^*_r(\Gamma,\sigma) \otimes \mathcal{K})/ \sim \) is an Abelian semi-group under direct sums, and the Grothendieck group \( K_0(C^*_r(\Gamma,\sigma)) \) is defined as the associated Abelian group.

Now the estimate on the number of equivalence classes of projections is achieved by computing the range of a trace. The von Neumann algebra \( \mathcal{U}(\Gamma,\sigma) \) and \( C^* \)-algebra \( C^*_r(\Gamma,\sigma) \) have a canonical faithful finite trace \( \tau \), where

\[
\tau(A) = \langle A\delta_1,\delta_1 \rangle_{\ell^2(\Gamma)},
\]

where \( \delta_j \) is the basis of \( \ell^2(\Gamma) \). If \( \text{Tr} \) denotes the standard trace on bounded operators in an \( \infty \)-dimensional separable Hilbert space \( \mathcal{H} \), then we obtain a trace

\[\text{tr} = \tau \otimes \text{Tr} : \text{Proj}(C^*_r(\Gamma,\sigma) \otimes \mathcal{K}) \to \mathbb{R}.\]

This induces a trace on the \( K \)-group

\[[\text{tr}] : K_0(C^*_r(\Gamma,\sigma)) \to \mathbb{R}\]

with

\[\text{tr}(\text{Proj}(C^*_r(\Gamma,\sigma))) = [\text{tr}](K_0(C^*_r(\Gamma,\sigma))) \cap [0,1].\]

The result quoted above in (11), counting the energy gaps in our hyperbolic model, can then be derived from the following result proved in [10].

**Theorem 2.1.** Let \( \Gamma \) be a cocompact Fuchsian group of signature \( (g : \nu_1,\ldots,\nu_n) \) and \( \sigma \) a multiplier on \( \Gamma \) with flux \( \theta \). Then the range of the trace is,

\[\text{tr}(K_0(C^*_r(\Gamma,\sigma))) = \mathbb{Z} + \theta\mathbb{Z} + \sum_j 1/\nu_j\mathbb{Z}.\]

Rieffel, and Pimsner and Voiculescu established analogous results in the case \( \Gamma = \mathbb{Z}^2 \). The result in the case of torsion-free Fuchsian groups was established in [8]. In more recent work, the second author generalized this result to discrete subgroups of rank 1 groups and to all amenable groups, and more generally whenever the Baum–Connes conjecture with coefficients holds for the discrete group, [12].

By contrast, the behavior of spectral gaps when the flux is irrational is still mysterious. The problem can be formulated in terms of the following conjecture (also known as the “generalized ten Martini problem”), cf. [10] [11].

**Conjecture 2.2.** Let \( \Gamma \) be a cocompact Fuchsian group and \( \sigma \) be a multiplier on \( \Gamma \). If the flux \( \theta \) is irrational, then there is a \( V \in \mathbb{C}(\Gamma,\sigma) \) such that \( H_{\sigma,V} \) has an infinite number of gaps in its spectrum.

It is not yet known if any gaps exist at all in this case! However, using Morse–type potentials, the second author and Shubin [13] proved that there is an arbitrarily large number of gaps in the
spectrum of magnetic Schrödinger operators on covering spaces, (i.e. in the continuous model).

Recent work of the second author with Dodziuk and Yates shows another interesting property of the spectrum, proving that all \( L^2 \) eigenvalues of the Harper operators of surface groups \( \Gamma \) are \textit{algebraic numbers}, whenever the multiplier is algebraic, that is, when \([\sigma] \in H^2(\Gamma, \mathbb{Q}/\mathbb{Z})\). In fact the same result remains true when adding potentials \( V \) in \( \mathbb{Q}(\Gamma, \sigma) \) to the Harper operator.

3 Hall conductance and cyclic cocycles

The main idea behind the geometric approach to the computation of the values of the Hall conductance (cf. [5], [4], [8], [11]) is to express the conductance, from physical considerations, in terms of the Connes–Kubo formula, as a certain cocycle evaluated on the spectral projection \( P_E \) that corresponds to the Fermi level \( E \), when \( E \) is in a gap of the spectrum. Since the values remain unchanged when the cocycle is modified in a certain equivalence class, it is then possible to replace the \textit{conductance cocycle} with a different but equivalent cocycle of a geometric origin, the \textit{area cocycle}, whose value on projections can be explicitly computed.

3.1 A smooth subalgebra

For several technical reasons, since we will need to use certain derivations in the definition of the Hall conductance cocycle, we shall only consider a dense \(*\)-subalgebra \( \mathcal{R} \) of the algebra of observables \( C^*_r(\Gamma, \sigma) \).

This subalgebra contains \( \mathbb{C}(\Gamma, \sigma) \) and is contained in the domain of definition of these derivations. It contains the spectral projection \( P_E \), when the Fermi level is in a gap of the energy spectrum. Moreover, it satisfies the following two key properties.

1. The inclusion \( \mathcal{R} \subset C^*_r(\Gamma, \sigma) \) induces an isomorphism in \( K\)-theory.

2. Polynomial growth group cocycles on \( \Gamma \) define cyclic cocycles on \( \mathbb{C}(\Gamma, \sigma) \) that extend continuously to \( \mathcal{R} \).

\( \mathcal{R} \) is defined as follows. Consider an operator \( D \) defined as

\[
D\delta_\gamma = \ell(\gamma)\delta_\gamma \forall \gamma \in \Gamma,
\]

where \( \ell(\gamma) \) denotes the word length of \( \gamma \). Let \( \delta = \text{ad}(D) \) denote the commutator \([D, \cdot]\). Then \( \delta \) is an unbounded, but closed derivation on \( C^*_r(\Gamma, \sigma) \). Define

\[
\mathcal{R} := \bigcap_{k \in \mathbb{N}} \text{Dom}(\delta^k).
\]

It is clear that \( \mathcal{R} \) contains \( \delta_\gamma \forall \gamma \in \Gamma \) and so it contains \( \mathbb{C}(\Gamma, \sigma) \). Hence it is dense in \( C^*_r(\Gamma, \sigma) \).

It is not hard to see that \( \mathcal{R} \) is closed under the holomorphic functional calculus, and therefore by a result of Connes, property (1) above holds, and by equation (12), \( P_E \in \mathcal{R} \). Until now, we have not used any special property of the group \( \Gamma \). But now assume that \( \Gamma \) is a surface group. Then it follows from a result by Jollisaint that there is a \( k \in \mathbb{N} \) and a positive constant \( C_0 \) such that for all \( f \in \mathbb{C}(\Gamma, \sigma) \), one has the \textit{Haagerup inequality},

\[
||f|| \leq C' \nu_k(f),
\]

(13)

where \( ||f|| \) denotes the operator norm of the operator on \( \ell^2(\Gamma) \) given by left convolution by \( f \). Using this, it is routine to show that property (2) holds. Note that the spectral projections onto gaps in the Hamiltonian \( H \) belong to the algebra of observables \( \mathcal{R} \), for any choice of electric potential \( V \).
3.2 Cyclic cocycles

Cyclic cocycles are also called multilinear traces, and the word cyclic refers to invariance under the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acting on the slots of the Cartesian product, i.e. $t$ is a cyclic $n$-cocycle if

$$t : \mathcal{R} \times \mathcal{R} \cdots \times \mathcal{R} \rightarrow \mathbb{C}$$

satisfies the cyclic condition

$$t(a_0, a_1, \ldots, a_n) = t(a_n, a_0, a_1, \ldots, a_{n-1}) = \ldots$$

$$= t(a_1, \ldots, a_n, a_0),$$

and satisfies the cocycle condition

$$t(aa_0, a_1, \ldots, a_n) - t(a, a_0a_1, \ldots, a_n) \ldots$$

$$(-1)^{n+1}t(a_0a, a_0, \ldots, a_{n-1}) = 0.$$

For instance, a cyclic 0-cocycle is just a trace. In fact, in this case the condition it satisfies is $t(ab) = t(ba)$. A cyclic 1-cocycle satisfies $t(a, b) = t(b, a)$ and $t(ab, c) - t(a, bc) + t(ca, b) = 0$, and a cyclic 2-cocycle satisfies

$$t(a, b, c) = t(c, a, b) = t(b, c, a)$$

and

$$t(ab, c, d) - t(a, bc, d) + t(a, b, cd) - t(da, b, c) = 0.$$

3.3 Conductance cocycle

A formula for the Hall conductance is obtained from transport theory. The following is a general mathematical formulation of the result.

Given a 1-cocycle $a$ on the discrete group $\Gamma$, i.e.

$$a(\gamma_1\gamma_2) = a(\gamma_1) + a(\gamma_2) \quad \forall \gamma_1, \gamma_2 \in \Gamma$$

one can define a linear functional $\delta_a$ on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$

$$\delta_a(f)(\gamma) = a(\gamma)f(\gamma)$$

Then one verifies that $\delta_a$ is a derivation:

$$\delta_a(fg)(\gamma) = a(\gamma)fg(\gamma)$$

$$= a(\gamma) \sum_{\gamma_1, \gamma_2} f(\gamma_1)g(\gamma_2)\sigma(\gamma_1, \gamma_2)$$

$$= \sum_{\gamma_1, \gamma_2} \left(a(\gamma_1) + a(\gamma_2)\right) f(\gamma_1)g(\gamma_2)\sigma(\gamma_1, \gamma_2)$$

$$= \sum_{\gamma_1, \gamma_2} \left(\delta_a(f)(\gamma_1)g(\gamma_2)\sigma(\gamma_1, \gamma_2) + f(\gamma_1)\delta_a(g)(\gamma_2)\sigma(\gamma_1, \gamma_2)\right)$$

$$= (\delta_a(f)g)(\gamma) + (f\delta_a(g))(\gamma).$$

In the case of a Fuchsian group $\Gamma$, the first cohomology of the group $\Gamma$ is a free Abelian group of rank $2g$, where $g$ is the genus of Riemann surface $\mathbb{H}/\Gamma$. It is in fact a symplectic vector space over $\mathbb{Z}$, and we can assume that $a_j, b_j, j = 1, \ldots, g$ is a symplectic basis for $H^1(\Gamma, \mathbb{Z})$.

We denote $\delta_{a_j}$ by $\delta_j$ and $\delta_{b_j}$ by $\delta_{j+g}$. These derivations give rise to a cyclic 2-cocycle on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$,

$$\text{tr}_K(f_0, f_1, f_2) =\sum_{j=1}^g \text{tr}(f_0(\delta_j(f_1))\delta_{j+g}(f_2) - \delta_{j+g}(f_1)\delta_j(f_2)))$$

$$\text{tr}_K$$ is called the conductance 2-cocycle

In more physical terms, one needs to determine the component of the induced current that is orthogonal to the applied potential, from which the conductivity is obtained by dividing by the magnitude of the applied field. By the effect of homogeneous effective interaction of all the charge carriers, which in our model corresponds to a single electron moving in hyperbolic geometry, the orthogonal direction now appears as split into a family of directions marked by the geodesics that correspond to the generators of the fundamental group of $\Gamma \setminus \mathbb{H}$, emanating from a given point in the hyperbolic plane $\mathbb{H}$. One interprets then the term

$$\text{tr}(f_0(\delta_j(f_1))\delta_{j+g}(f_2) - \delta_{j+g}(f_1)\delta_j(f_2)))$$

(15)
as computing, in the quantum adiabatic limit [14], the conductivity for currents in the \((j + g)\)-th direction induced by electric fields in the \(j\)-th direction.

Since \(a_j, b_j\) are (polynomially) bounded, it follows that the conductance 2-cocycle extends to a cyclic 2-cocycle on \(\mathcal{R}\).

Because the charge carriers are Fermions, two different charge carriers must occupy different quantum eigenstates of the Hamiltonian \(H\). In the limit of zero temperature they minimize the energy and occupy eigenstates with energy lower that a given one, called the Fermi level and denoted by \(E\). Let \(P_E\) denote denote the corresponding spectral projection, \(i.e.\ P_E = \chi_{(-\infty,E]}(H)\). Then the Hall conductance is given by

\[
\sigma_E = \text{tr}_K(P_E, P_E, P_E).
\]

### 3.4 The quantum adiabatic limit

We recall briefly the justification of (15) in terms of the quantum adiabatic limit for a slowly varying time dependent Hamiltonian, \(cf.\ [15]\).

If \(H(s)\) is a smooth family of self-adjoint Hamiltonians and \(P(s)\) are spectral projections on a gap in the spectrum of \(H(s)\), then

\[
X(s) = \frac{1}{2\pi i} \oint_C R(z, s) \partial_s P(s) R(z, s) dz,
\]

with \(R(z, s) = (H(s) - z)^{-1}\), satisfies the commutation relations

\[
[\partial_s P(s), P(s)] = [H(s), X(s)].
\]

The quantum adiabatic limit theorem (\(cf.\ [14]\)) then shows that the adiabatic evolution approximates well the physical evolution, for large values of the adiabatic parameter \(\tau \to \infty\), via an estimate of the form

\[
\| (U_\tau(s) - U_a(s)) P(0) \| \lesssim \frac{1}{\tau} \max_{s \in [0, \infty)} \{2 \| X(s) P(s) \| + \| \partial_s (X(s) P(s)) P(s) \| \}.
\]

Here the physical evolution satisfies

\[
i \partial_s U_\tau(s) = \tau H(s) U_\tau(s),
\]

\(U_\tau(0) = 1\), where \(s = t/\tau\) is a scaled time, and the adiabatic evolution is defined by the equation

\[
P(s) = U_a(s) P(0) U_a(s)^*\]

with \(U_a(0) = 1\).

In our setting, the functional derivative \(\delta_k H\), with respect to a component \(A_k\) of the magnetic potential, gives a current density \(J_k\). Its expectation value in a state described by a projection \(P\) on a gap in the spectrum of the Hamiltonian is then computed by \(\text{tr}(P \delta_k H)\). In the quantum adiabatic limit, one can replace \(\delta_k H\) with \(\delta_k H_a\), where the adiabatic Hamiltonian \(H_a\) satisfies

\[
i \partial_s U_a(s) = \tau H_a(s) U_a(s)
\]

and the equation of motion

\[
[H_a(s), P(s)] = \frac{i}{\tau} \partial_s P(s).
\]

This implies that the relation

\[
\text{tr}(P[\partial_t P, \delta_k P]) = \text{itr}(\delta_k (PH_a)) - \text{itr}(P \delta_k H_a).
\]

We make some simplifying assumptions. If the trace is invariant under variations of \(A_k\), then the first term in the right hand side of (16) vanishes. We also assume that the only time dependence of \(H\) and \(P\) is in the adiabatic variation of a component \(A_j\) distinct from \(A_k\), and we work in the Landau gauge, so that the electrostatic potential venishes and the electric field is given by \(E = -\partial A / \partial t\). Then we have \(\partial_t = -E_j \partial_j\), so that the expectation of the current \(J_k\) is given by

\[
\text{tr}(P \delta_k H) = \text{itr}(P[\partial_t P, \delta_k P])
\]
hence the conductance for a current in the $k$ direction induced by an electric field in the $j$ direction is given by $-iE_j \text{tr}(P[\delta_j P, \delta_k P])$. The analytic aspects of this formal argument can be made rigorous following the techniques used in [6].

### 3.5 Area cocycle

Our goal is to prove that the conductance cyclic 2-cocycle is fractional, i.e., it takes on fractional values on projections in the dense subalgebra $\mathcal{R}$ of the algebra of observables. The technical role played by $\mathcal{R}$ is that it has the same cyclic cohomology as the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$ and the same $K$-theory as $C^*(\Gamma, \sigma)$.

More explicitly, we want to replace the conductance cocycle by another 2-cocycle, which takes the same values on projections in $\mathcal{R}$. We say that two cyclic 2-cocycles $t_1$ and $t_2$ differ by a coboundary (that is, they define the same cyclic cohomology class) if

$$t_1(a_0, a_1, a_2) - t_2(a_0, a_1, a_2) = \lambda(a_0a_1, a_2) - \lambda(a_0, a_1a_2) + \lambda(a_2a_0, a_1),$$

where $\lambda$ is a cyclic 1-cocycle.

The second cyclic 2-cocycle is the area 2-cocycle, which is obtained as follows. On $G = \text{PSL}(2, \mathbb{R})$, there is an (area) 2-cocycle

$$C : G \times G \to \mathbb{R}$$

$$C(\gamma_1, \gamma_2) = \text{(oriented) hyperbolic area of the geodesic triangle with vertices at} (0, \gamma_1^{-1}o, \gamma_2o), \quad o \in \mathbb{H}$$

The restriction of this to a discrete subgroup $\Gamma$ gives the area group cocycle on $\Gamma$. This in turn defines a cyclic 2-cocycle on $C(\Gamma, \sigma)$ by

$$\text{tr}_C(f_0, f_1, f_2) = \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} f_0(\gamma_0)f_1(\gamma_1)f_2(\gamma_2)C(\gamma_1, \gamma_2)\sigma(\gamma_1, \gamma_2).$$

(17)

Since $C$ is (polynomially) bounded, $\text{tr}_C$ can be shown to extend to the smooth subalgebra $\mathcal{R}$.

We can compare the conductance and the area cyclic cocycles, through the following result of [8] [11].

**Proposition 3.1.** The cocycles $\text{tr}_K$ and $\text{tr}_C$ differ by a coboundary, which can be described explicitly. Since they are cohomologous, $\text{tr}_K$ and $\text{tr}_C$ induce the same map on $K$-theory.

In fact, the difference between the conductance cocycle $\text{tr}_K$ and the area cocycle $\text{tr}_C$ can be evaluated in terms of the difference between the hyperbolic area of a geodesic triangle and the Euclidean area of its image under the Abel-Jacobi map, cf. [11]. This difference can be expressed as a sum of three terms

$$U(\gamma_1, \gamma_2) = h(\gamma_2^{-1}, 1) - h(\gamma_1^{-1}, \gamma_2) + h(1, \gamma_1),$$

where each integral is a difference of line integrals, one along a geodesic segment in $\mathbb{H}$ and one along a straight line in the Jacobian variety. The cocycles correspondingly differ by

$$\text{tr}_K(f_0, f_1, f_2) - \text{tr}_C(f_0, f_1, f_2) = \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} f_0(\gamma_0)f_1(\gamma_1)f_2(\gamma_2)U(\gamma_1, \gamma_2)\sigma(\gamma_1, \gamma_2).$$

This expression can be written as $\lambda(f_0f_1, f_2) - \lambda(f_0, f_1f_2) + \lambda(f_2f_0, f_1)$ where $\lambda(f_0, f_1) = \sum_{\gamma_0 \gamma_1 = 1} f_0(\gamma_0)f_1(\gamma_1)h(1, \gamma_1)\sigma(\gamma_0, \gamma_1).$
3.6 The index formula

The problem of deriving the values of the Hall conductance is now reduced to computing the pairing of the area cyclic 2-cocycle with K-theory.

The following key result, proved in [11], is established by proving a twisted analogue of the Baum–Connes conjecture for the group $\Gamma$ and also a modest generalization of the higher index theorem of Connes–Moscovici [16].

**Theorem 3.2.** Let $\Gamma$ be a cocompact Fuchsian group of signature $(g : \nu_1, \ldots, \nu_n)$ of genus $g > 1$. Then one has

$$[\text{tr}_K](K_0(C^*_r(\Gamma, \sigma))) = [\text{tr}_C](K_0(C^*_r(\Gamma, \sigma))) = \phi \mathbb{Z},$$

where $\phi = (2(g - 1) + (n - \sum_j 1/\nu_j))$ is the orbifold Euler characteristic of $\Gamma \setminus \mathbb{H}$. That is, the conductance 2-cocycle $[\text{tr}_K]$ is an integer multiple of the orbifold Euler characteristic, and is in particular fractional.

The actual computation of $\phi$ is done via characteristic classes, where we observe that $\phi$ is proportional to the hyperbolic orbifold volume.

The $K$-theory $K_0(C^*_r(\Gamma, \sigma))$ is also naturally the receptacle for the analytic index of elliptic operators of the form $D^+_E \otimes \nabla$, where $D^+_E$ is the Dirac operator acting on spinors in $S^+ \otimes E$, where $E$ is an orbifold vector bundle over $\Gamma \setminus \mathbb{H}$, and $\nabla = d + iA$ is a Hermitian connection with curvature the exact $\Gamma$-invariant 2-form $B$ describing the magnetic field.

Applying a version of the results of [16] that accounts for the presence of the magnetic field (in the form of the multiplier $\sigma$), and the pairing between cyclic cohomology and $K$-theory of [17], we obtain an index pairing that depends on a group cocycle $c \in H^2(\Gamma)$, and on the magnetic field, via the multiplier $\sigma$.

$$\text{Ind}_{c, \Gamma, \sigma}(\rho^+_E \otimes \nabla) = \frac{1}{2\pi|G|} \int_{\Sigma_{g'}} \hat{A} \text{ tr}(e^{R_E}) e^B \hat{c}. $$

We explain the terms that appear in the integral: $(\Sigma_{g'}, G)$ is a smooth orbifold cover of $\Sigma(g; \nu_1, \ldots, \nu_n) = \Gamma \setminus \mathbb{H}$ by a smooth Riemann surface of genus $g'$ and with $|G|$ the order of the finite group acting on $\Sigma_{g'}$. The form $\hat{A} \text{ tr}(e^{R_E})$ is the usual Chern–Weil form for the index of the twisted Dirac operator $\rho^+_E$ and $iB = \nabla^2$ is the 2-form of the magnetic field. Moreover, $\hat{c}$ is the lift to a 2-form on $\Sigma_{g'}$ of $c$ regarded as an element in the orbifold cohomology of $\Sigma(g; \nu_1, \ldots, \nu_n)$.

The form $\hat{A} \text{ tr}(e^{R_E}) e^B \hat{c}$ is expressed as a power series, but since we integrate over the 2-dimensional surface $\Sigma_{g'}$, we only need to consider the terms up to order 2, hence the index formula simplifies to

$$\text{Ind}_{c, \Gamma, \sigma}(\rho^+_E \otimes \nabla) = \frac{\text{rank}_E}{2\pi|G|} \int_{\Sigma_{g'}} \hat{c}. \quad (18)$$

What is remarkable about (18) is that it isolates the dependence on the magnetic field in the form of the orbifold bundle $E$ associated to the spectral projection $P_E$ of the Hamiltonian $H_{\sigma, V}$, and that it reduces the computation of the conductance $\sigma_E$ to the evaluation of just one integral, $\int_{\Sigma_{g'}} \hat{c}$.

3.7 Orbifold Euler characteristic and fractions

For any compact oriented 2-dimensional orbifold $\Sigma(g; \nu_1, \ldots, \nu_n)$, with $n \geq 3$, or $n = 2$ and $\nu_1 \neq \nu_2$ in the case $g = 0$, there exists a (non-unique) pair $(\Sigma_{g'}, G)$ of a smooth compact surface $\Sigma_{g'}$ of genus $g'$ with no marked
points and a finite group $G$ acting on $\Sigma_{g'}$ by isometries, with quotient $\Sigma(g; \nu_1, \ldots, \nu_n)$ (good orbifold). The data $(\Sigma_{g'}, G)$ correspond to a group extension $1 \to \Gamma_{g'} \to \Gamma \to G \to 1$, with $\Gamma_{g'}$ the Fuchsian group uniformizing $\Sigma_{g'}$, and are subject to the constraint $2(g' - 1)/|G| = 2(g - 1) + (n - \nu)$, with $\nu = \sum_{j=1}^{n} 1/\nu_j$. The value $\chi_{orb} = 2(1 - g) + (\nu - n)$ is an important rational valued invariant of orbifold geometry, the orbifold Euler characteristic.

In order to compute the values of the Hall conductance, we only need to evaluate the integral $\int_{\Sigma_{g'}} c_A$, where $c_A$ is the area cocycle. This turns out to be (cf. [11] [8]) the integration of the volume form, namely the hyperbolic volume of $\Sigma_{g'}$,

$$\int_{\Sigma_{g'}} c_A = 2\pi(2g' - 2),$$

where here we use the Gauss-Bonnet theorem for hyperbolic surfaces. Thus, we have obtained

$$\sigma_E = \frac{(2g' - 2)\text{rank}\mathcal{E}}{|G|},$$

that is,

$$\sigma_E = -\chi_{orb}(\Sigma(g; \nu_1, \ldots, \nu_n)) \text{rank}\mathcal{E},$$

and we conclude that the Hall conductance is quantized at integer multiples of the rational quantities $\chi_{orb}(\Sigma(g; \nu_1, \ldots, \nu_n))$. We have obtained the following statement.

**Corollary 3.3.** Suppose that the Fermi level $E$ lies in a spectral gap of the Hamiltonian $H_{\sigma,V}$. Then the Hall conductance

$$\sigma_E = \text{tr}_K(P_E, P_E, P_E) = \text{tr}_C(P_E, P_E, P_E) \in \phi\mathbb{Z},$$

i.e. the Hall conductance has plateaux that are integer multiples of the orbifold Euler characteristic $\phi = -\chi_{orb}(\Gamma \backslash \mathbb{H}).$

4 Discussion

A key advantage of our hyperbolic model is that it treats the FQHE within the same framework developed by Bellissard et al. for the IQHE, with hyperbolic geometry replacing Euclidean geometry, to account for the effect of electron correlation, while remaining formally within a single particle model.

The fractions for the Hall conductance that we get are obtained from an equivariant index theorem and are thus topological in nature. Consequently, the Hall conductance is seen to be stable under small deformations of the Hamiltonian. Thus, this model can be generalized to systems with disorder as in [18], and then the hypothesis that the Fermi level is in a spectral gap of the Hamiltonian can be relaxed to the assumption that it is in a gap of extended states. This is a necessary step in order to establish the presence of plateaux.

In fact, this solves the apparent paradox that we still have a FQHE, even though the Hamiltonian $H_{\sigma,V}$ may not have any spectral gaps. The reason is that, as explained in [18], the domains of the cyclic 2-cocycles $\text{tr}_C$ and $\text{tr}_K$ are in fact larger than the smooth subalgebra $\mathcal{R}$. More precisely, there is a *-subalgebra $\mathcal{A}$ such that $\mathcal{R} \subset \mathcal{A} \subset \mathcal{U}(\Gamma, \sigma)$ and $\mathcal{A}$ is contained in the domains of $\text{tr}_C$ and $\text{tr}_K$. $\mathcal{A}$ is closed under the Besov space functional calculus, and the spectral projections $P_E$ of the Hamiltonian $H_{\sigma,V}$ that lie in $\mathcal{A}$ are called gaps in extended states. They include all the spectral projections onto gaps in the energy spectrum, but contain many more spectral projections. In particular, even though the Hamiltonian $H_{\sigma,V}$ may not have any spectral gaps, it may still have gaps in extended states. The results extend in a straightforward way to the case with disorder, where one allows
the potential $V$ to be random, cf. [18].

In this model we can easily obtain most of the experimentally observed fractions. As the first table illustrates, low genus orbifolds with a small number of cone points are sufficient to recover many observed fractions. What appears very promising is the fact that the fractions that are more easily observed experimentally, i.e. those that appear with a larger and more clearly marked plateau (cf. e.g. [19], [20]), also correspond to orbifold Euler characteristics that are realized by a large number of orbifolds.

The main limitation of our model is that it seems to predict too many fractions, which at present do not seem to correspond to experimentally observed values. To our knowledge, however, this is also a limitation in the other theoretical models available in the literature. Another limitation is the fact that this model does not explain why even denominator fractions are more difficult to observe than odd ones. In fact, even for small number of cone points and low genus, one obtains a large number of orbifold Euler characteristics with even denominator, which are not justified experimentally. On the occurrence of even denominators in the fractional quantum Hall effect experiments, cf. e.g. [21] [22] [23].

A prediction of our model of FQHE is the existence of an absolute lower bound on the fractional values of the Hall conductance. The lower bound is imposed by the orbifold geometry, and does not have an analog in other theoretical models, hence it appears to be an excellent possible experimental test of the validity of our theoretical model. The lower bound is obtained from the Hurwitz theorem, which states that the maximal order of a finite group $G$ acting by isometries on a smooth Riemann surface $\Sigma_{g'}$ is $\#G = 84(g' - 1)$. This imposes the constraint

$$\phi \geq \frac{2(g' - 1)}{84(g' - 1)} = \frac{1}{42}.$$  

The lower bound is realized by $1/42 = -\chi_{orb}(\Sigma(0; 2, 3, 7)).$

### 4.1 Tables

In the first table, we consider experimentally observed fractions, which we recover in our model. Notice how fractions like $1/3$, $2/5$, $2/3$, which experimentally appear with a wider and more clearly marked plateau, also correspond to the fractions realized by a larger number of orbifolds (we only checked the number of solutions for small values $\nu_j \leq 20$, $n = 3$, $g = 0$, and $\phi < 1$). These observations should be compared with the experimental data, cf. e.g. [24] [20].

The second table shows how to obtain some experimentally observed fractions with $\phi > 1$ (without counting multiplicities).

The third table contains a list of further fractions with odd denominators that are predicted by our model, while the fourth table contains a list of some even denominator fractions predicted by our model.
### Experimental Results

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### Experimental Results

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References


