Towards the fractional quantum Hall effect: a noncommutative geometry perspective

Based on

- M.Marcolli and V.Mathai, *Twisted index theory on good orbifolds, II: fractional quantum numbers*, Communications in Mathematical Physics, Vol.217, no.1 (2001) 55-87.

- M.Marcolli and V.Mathai, *Twisted index theory on good orbifolds, I: noncommutative Bloch theory*, Communications in Contemporary Mathematics, Vol.1 (1999) 553-587.

Electrons in solids – Bloch theory

Crystals Bravais lattice $\Gamma \subset \mathbb{R}^d$ $(d = 2, 3)$

Periodic potential (electron–ions interaction)

$$U(x) = \sum_{\gamma \in \Gamma} u(x - \gamma) \qquad T_\gamma U = U, \forall \gamma \in \Gamma$$

Mutual interaction of electrons

$$\sum_{i=1}^{N} (-\Delta_{x_i} + U(x_i)) + \frac{1}{2} \sum_{i \neq j} W(x_i - x_j)$$

Independent electron approximation

$$\sum_{i=1}^{N} (-\Delta_{x_i} + V(x_i))$$

modification V of single electron potential

$\psi(x_1, \ldots, x_N) = \det(\psi_i(x_j))$ for $(-\Delta + V)\psi_i = E_i \psi_i$ so $\sum(-\Delta_{x_i} + V(x_i))\psi = E\psi$ with $E = \sum E_i$ (reduction to single electron)

Inverse problem: determine V

Bloch electrons

T_γ = unitary operator on $\mathcal{H} = L^2(\mathbb{R}^d)$ translation by $\gamma \in \Gamma$

$$T_\gamma \, H \, T_{\gamma^{-1}} = H \quad \forall \gamma \in \Gamma$$

for $H = -\Delta + V$

Simultaneously diagonalize: characters of $\Gamma \Rightarrow$ Pontrjagin dual $\hat{\Gamma}$

$T_\gamma \psi = c(\gamma)\psi$, with $c : \Gamma \to U(1)$ since $T_{\gamma_1\gamma_2} = T_{\gamma_1} T_{\gamma_2}$

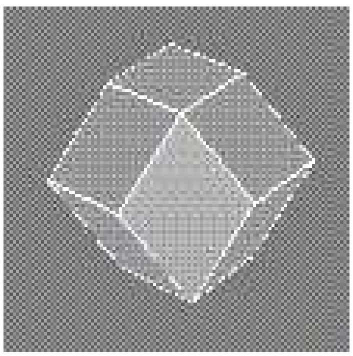
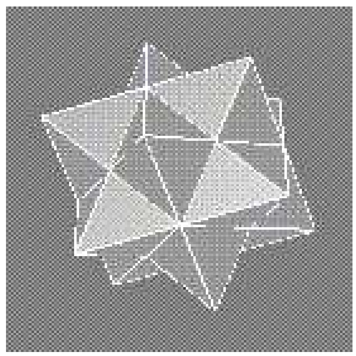
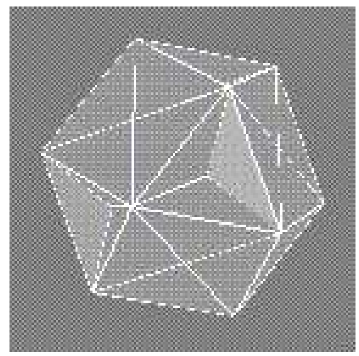
$$c(\gamma) = e^{i\langle k,\gamma\rangle}, \qquad k \in \hat{\Gamma}$$

$\hat{\Gamma}$ compact group isom to T^d (dual of $\mathbb{R}^d$ mod reciprocal lattice $\Gamma^\sharp$

$$\Gamma^\sharp = \{k \in \mathbb{R}^d : \langle k, \gamma\rangle \in 2\pi\mathbb{Z}, \forall \gamma \in \Gamma\}$$

Brillouin zones

Fundamental domains of $\Gamma^\sharp$: n-th zone, points such that line to the origin crosses exactly $(n-1)$ Bragg hyperplanes of the crystal

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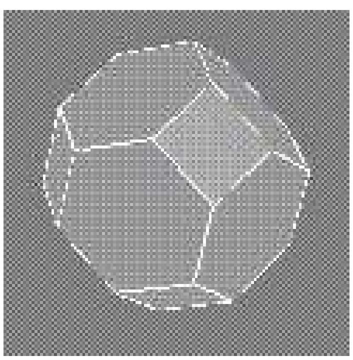
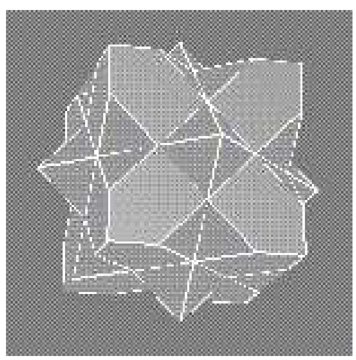
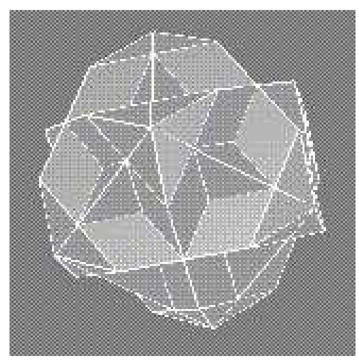

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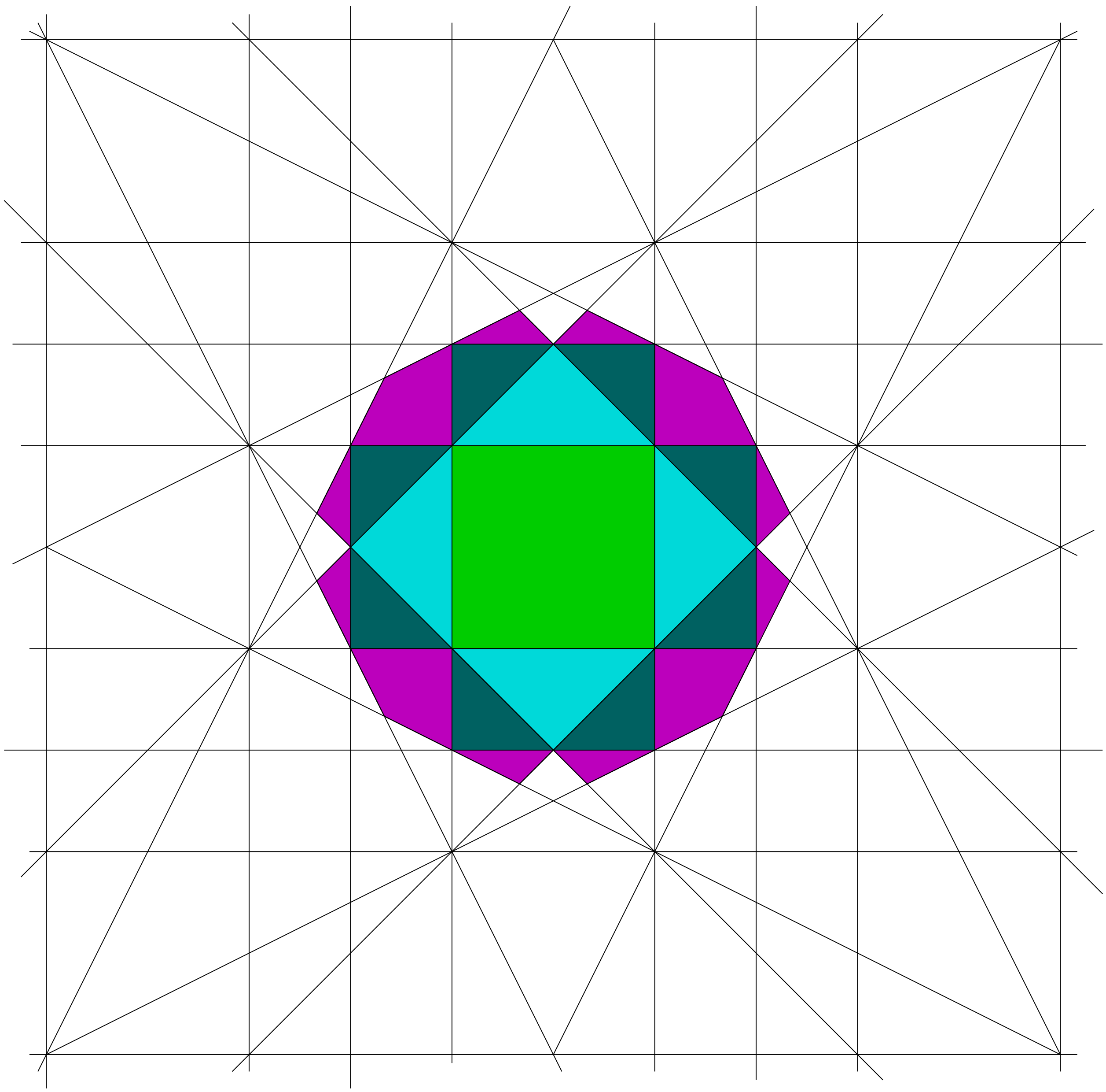


Band structure

Self-adjoint elliptic boundary value problem

$$D_k = \left\{ \begin{array}{c} (-\Delta + V)\psi = E\psi \\ \psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\psi(x) \end{array} \right.$$

Eigenvalues $\{E_1(k), E_2(k), E_3(k), \ldots\}$

$$E(k) = E(k+u) \quad \forall u \in \Gamma^\sharp$$

Plot $E_n(k)$ over the n-the Brillouin zone

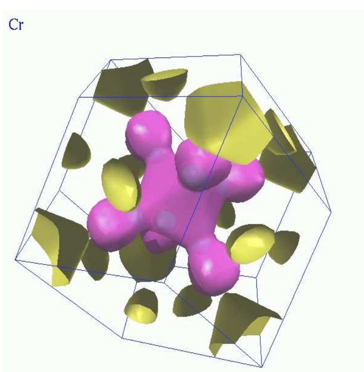
$$k \mapsto E(k) \quad k \in \mathbb{R}^d$$

energy–crystal momentum dispersion relation

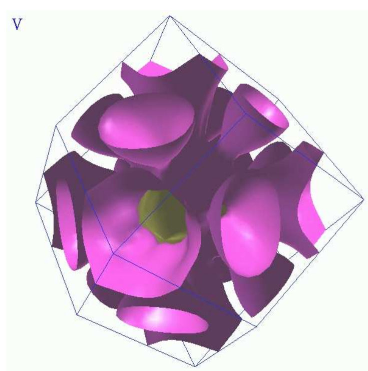
Fermi surface

Surface F in space of crystal momenta k determines electric and optical properties of the solid

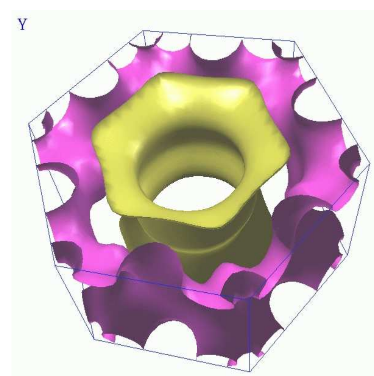
$$F_\lambda(\mathbb{R}) = \{k \in \mathbb{R}^d : E(k) = \lambda\}$$



Chromium



Vanadium



Yttrium

Complex Bloch variety

$$B(V) = \left\{ (k,\lambda) \in \mathbb{C}^{d+1} : \begin{array}{c} \exists\psi \ \text{ nontriv sol of} \\ (-\Delta + V)\psi = \lambda\psi \\ \psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\psi(x) \end{array} \right\}$$

$$F_\lambda(\mathbb{C}) = \pi^{-1}(\lambda) \subset B(V)$$

complex hyperfurface in $\mathbb{C}^d \Rightarrow$ singular projective

$F_\lambda \cap \mathbb{R}^d = F_\lambda(\mathbb{R})$ cycle in $H_{d-1}(F_\lambda(\mathbb{C}), \mathbb{Z})$

Integrated density of states

$$\rho(\lambda) = \lim_{\ell\to\infty} \frac{1}{\ell} \#\{\text{eigenv of } H \leq \lambda\}$$

$H = -\Delta + V$ on $L^2(\mathbb{R}^d/\ell\Gamma)$

Period

$$\frac{d\rho}{d\lambda} = \int_{F_\lambda(\mathbb{R})} \omega_\lambda$$

$\omega_\lambda =$ holom differential on $F_\lambda(\mathbb{C})$

Discretization on $\ell^2(\mathbb{Z}^d)$ Random Walk

$$\begin{aligned}\mathcal{R}\psi(n_1,\ldots,n_d) = \; & + \textstyle\sum_{i=1}^d \psi(n_1,\ldots,n_i+1,\ldots,n_d) \\ & + \textstyle\sum_{i=1}^d \psi(n_1,\ldots,n_i-1,\ldots,n_d)\end{aligned}$$

Discretized Laplacian

$$\Delta\psi(n_1,\ldots,n_d) = (2d-\mathcal{R})\;\psi(n_1,\ldots,n_d)$$

Bloch variety (polynomial equation in z_i, z_i^{-1})

$$B(V) = \left\{ (z_1\ldots,z_d,\lambda) : \begin{array}{l} \exists\psi\in\ell^2(\Gamma) \text{ nontriv sol of} \\ (\mathcal{R}+V)\psi = (\lambda+2d)\,\psi \\ R_{\gamma_i}\psi = z_i\,\psi \end{array} \right\}$$

$R_{\gamma_i}\psi(n_1,\ldots,n_d) = \psi(n_1,\ldots,n_i+a_i,\ldots,n_d)$

Random walk for discrete groups on $\mathcal{H} = \ell^2(\Gamma)$

Symmetric set of generators γ_i of Γ

$$\mathcal{R}\,\psi(\gamma) = \sum_{i=1}^r R_{\gamma_i}\psi\,(\gamma) = \sum_{i=1}^r \psi(\gamma\gamma_i)$$

Discretized Laplacian $\Delta = r - \mathcal{R}$

The breakdown of classical Bloch theory

- Magnetic field

- Aperiodicity

In both cases $T_\gamma H = H T_\gamma$ fails
$\Rightarrow$ noncommutativity

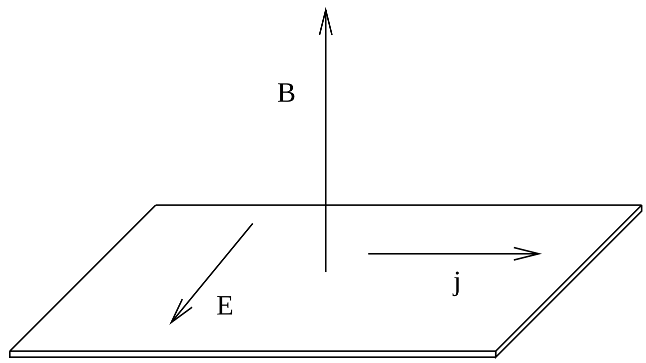
J.Bellissard, A.van Elst, H.Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect*, J.Math.Phys. 35 (1994) 5373–5451

Noncommutative Bloch theory

Integer Quantum Hall Effect

Hall Effect (Classical 1879)

thin metal, orthogonal strong magnetic field $\mathbf{B}$, current $\mathbf{j} \Rightarrow$ electric field $\mathbf{E}$, transverse Hall current



equilibrium of forces $\Rightarrow$ linear relation

$$Ne\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = 0$$

Hall conductance

$$\sigma_H = \frac{Ne\delta}{B}$$

intensity of Hall current/intensity of magnetic field

$$\sigma_H = \frac{\nu}{R_H}$$

filling factor=density of charge carriers $\cdot \frac{h}{eB}$

Hall resistance $R_H = h/e^2$ universal constant (fine structure constant $e^2/\hbar c$)

Integer Quantum Hall Effect

(von Klitzing 1980)

low temperature ($\leq 1K$), strong magnetic field

- σ_H as function of ν has plateaux at integer multiples of e^2/h

- At values of ν corresponding to plateaux, conductivity along current density axis vanishes

(precision 10^{-8}, independent of different samples, geometries, impurities)

Laughlin (1981): geometric origin
Thouless et al. (1982); Avron, Seiler, Simon (1983): topological (Gauss–Bonnet)
Bellissard et al. (1994): noncommutative geometry (explains also vanishing of direct conductivity)

Magnetic field

2-form $\omega = d\eta$ (field and potential $\mathbf{B} = \text{curl}\mathbf{A}$)

Magnetic Schrödinger operator

$$-(\nabla - i\eta)^2 + V$$

magnetic Laplacian $\Delta^\eta := (d - i\eta)^* \, (d - i\eta)$

Periodicity: $\gamma^*\omega = \omega$ (e.g. constant field) $0 = \omega - \gamma^*\omega = d(\eta - \gamma^*\eta)$

$$\eta - \gamma^*\eta = d\phi_\gamma$$

Magnetic Laplacian no longer commutes with Γ action

What symmetries?

Magnetic translations

$\phi_\gamma(x) = \int_{x_0}^{x} (\eta - \gamma^* \eta)$

$$T_\gamma^\phi \psi := \exp(i\phi_\gamma) T_\gamma \psi$$

then $(d - i\eta) T_\gamma^\phi = T_\gamma^\phi (d - i\eta)$

Magnetic translations no longer commute

$$T_\gamma^\phi T_{\gamma'}^\phi = \sigma(\gamma, \gamma') T_{\gamma\gamma'}^\phi$$

projective representation of Γ

$$\sigma(\gamma, \gamma') = \exp(-i\phi_\gamma(\gamma' x_0))$$

$\phi_\gamma(x) + \phi_{\gamma'}(\gamma x) - \phi_{\gamma'\gamma}(x)$ indep of x

(except integer flux case, where commute)

Pontrjagin dual vs. group C^*-algebra

Γ discrete abelian $\Leftrightarrow$ $\hat{\Gamma}$ compact abelian

Pontrjagin dual $\quad e^{i\langle k,\gamma\rangle} \quad$ characters

Algebra of functions

$$C(\hat{\Gamma}) \cong C_r^*(\Gamma)$$

C^*-algebra generated by Γ regular representation on

$$\ell^2(\Gamma) = \{\psi : \Gamma \to \mathbb{C} : \sum_\gamma |\psi(\gamma)|^2 < \infty\}$$

Fourier transform

When Γ non-abelian, $C_r^*(\Gamma)$ still makes sense (non-abelian)

$\hat{\Gamma}$ exists as a noncommutative space

Magnetic field $\Rightarrow$ Brillouin zone becomes noncommutative space

Harper operator (discretized magnetic Laplacian)

$$\begin{aligned} H_{\alpha_1,\alpha_2}\psi(m,n) = \; & e^{-i\alpha_1 n} && \psi(m+1,n) \\ + \; & e^{i\alpha_1 n} && \psi(m-1,n) \\ + \; & e^{-i\alpha_2 m} && \psi(m,n+1) \\ + \; & e^{i\alpha_2 m} && \psi(m,n-1) \end{aligned}$$

Let $\sigma((m',n'),(m,n)) = \exp(-i(\alpha_1 m' n + \alpha_2 m n'))$
(2-cocycle $\sigma : \Gamma \times \Gamma \to U(1)$)
Magnetic translations $U = R^{\sigma}_{(0,1)}$, $V = R^{\sigma}_{(1,0)}$

$$U\psi(m,n) = \psi(m,n+1)e^{-i\alpha_2 m}$$

$$V\psi(m,n) = \psi(m+1,n)e^{-i\alpha_1 n}$$

$\Rightarrow$ Noncommutative torus ($\theta = \alpha_2 - \alpha_1$)

$$UV = e^{i\theta}VU$$

Harper operator $H_\sigma = U + U^* + V + V^*$

Harper operators for discrete groups

multiplier $\sigma : \Gamma \times \Gamma \to U(1)$ (2-cocycle)

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)$$

$\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$

Hilbert space: $\ell^2(\Gamma)$

Left/right σ-regular representations

$$L^\sigma_\gamma \psi(\gamma') = \psi(\gamma^{-1}\gamma')\sigma(\gamma, \gamma^{-1}\gamma')$$

$$R^\sigma_\gamma \psi(\gamma') = \psi(\gamma'\gamma)\sigma(\gamma', \gamma)$$

satisfy

$$L^\sigma_\gamma L^\sigma_{\gamma'} = \sigma(\gamma, \gamma')L^\sigma_{\gamma\gamma'} \qquad R^\sigma_\gamma R^\sigma_{\gamma'} = \sigma(\gamma, \gamma')R^\sigma_{\gamma\gamma'}$$

Harper operator

(symmetric set of generators $\{\gamma_i\}_{i=1}^r$ of Γ)

$$\mathcal{R}_\sigma = \sum_{i=1}^r R^\sigma_{\gamma_i}$$

Twisted group algebras

$\mathbb{C}(\Gamma,\sigma)$ algebra of observables (magnetic translations), rep in $\mathcal{B}(\ell^2(\Gamma))$ by right σ-regular rep R^σ_γ

- weak closure $\mathcal{U}(\Gamma,\sigma)$ twisted group von Neumann algebra
- norm closure $C^*_r(\Gamma,\sigma)$ twisted (reduced) group C^*-algebra

Integer QHE case

$\Gamma = \mathbb{Z}^2$ (σ and θ as above)

$$C^*_r(\Gamma,\sigma) \cong A_\theta$$

irrational rotation algebra, NC torus

Connes-Chern character $\Rightarrow$ integer quantization of Hall conductance

Spectral theory

Magnetic Schrödinger $\mathcal{R}^\sigma + V$

potential $V \in \mathbb{C}(\Gamma, \sigma)$

Self–adjoint bounded $\Rightarrow$ Spec$(\mathcal{R}^\sigma + V) \subset \mathbb{R}$

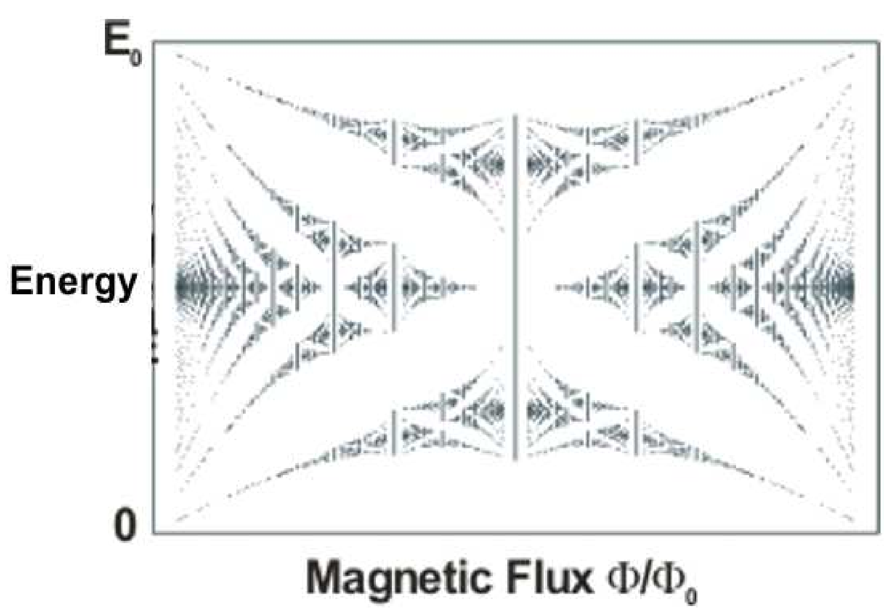
complement = collection of intervals

- Finitely many (Band structure, gaps)
- Infinitely many (Cantor-like spectrum)

For $\Gamma = \mathbb{Z}^2$, depends on rationality/irrationality of flux

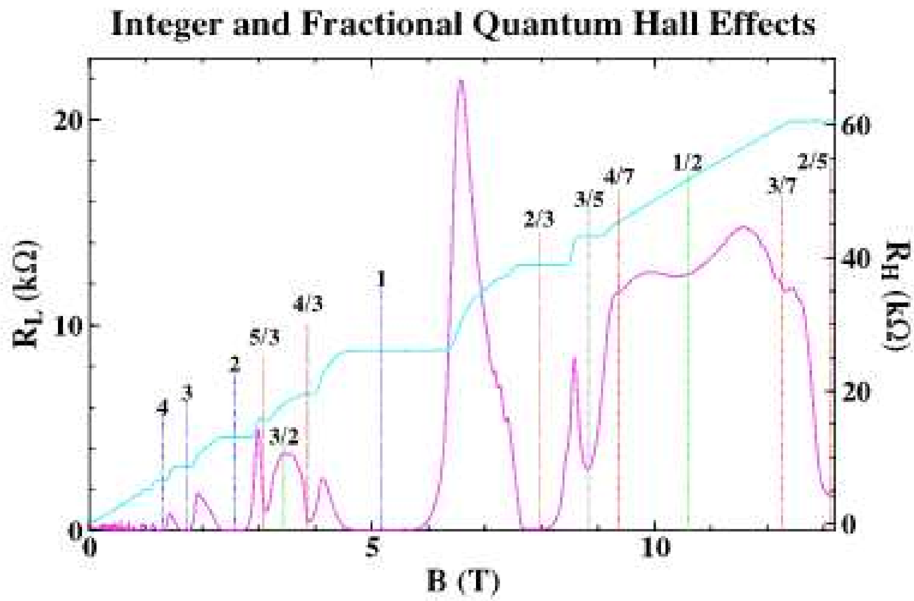
$$\Phi = \langle [\sigma], [\Gamma] \rangle$$

Hofstadter butterfly



Fractional Quantum Hall Effect

(Stormer and Tsui, 1983)



(high quality semi-conductor interface, low carrier concentration, and extremely low temperatures $\sim 10mK$, strong magnetic field)

- Plateaux at certain rational multiples of e^2/h

- Strongly interacting electrons

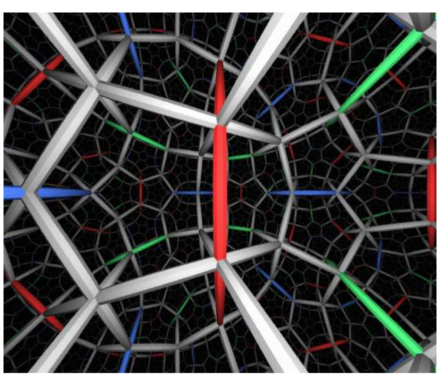
Noncommutative geometry model?

What is expected of such model?

- Account for strong electron interactions
- Exhibit observed fractions (+predictions)
- Account for varying width of plateaux

Simulate interaction via curvature

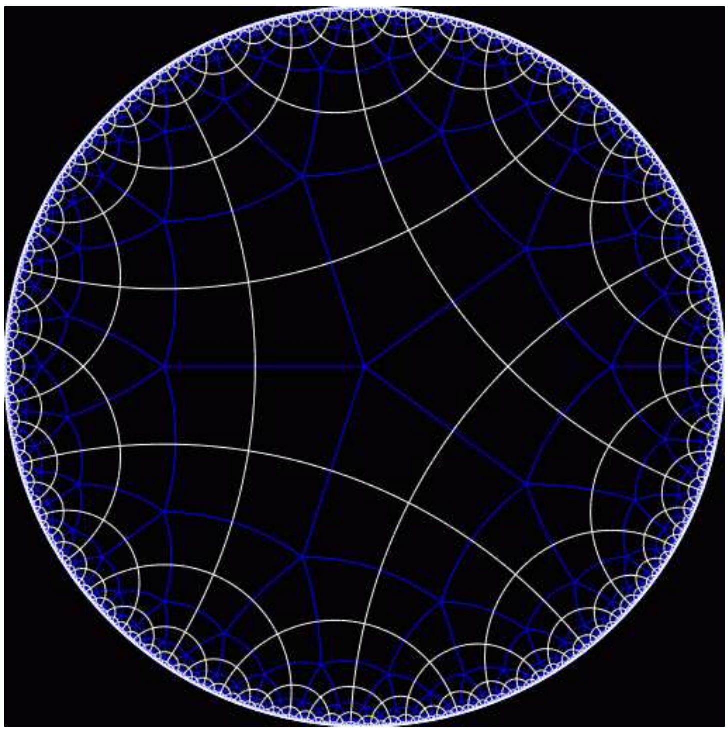
single electron in curved geometry “as if” subject to an average strong multi-electron interaction



Hall effect in the hyperbolic plane

Cocompact Fuchsian group $\Gamma = \Gamma(g, \nu_1, \ldots, \nu_n)$

$\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ discrete cocompact, genus g, elliptic elements order $\nu_1, \ldots, \nu_n$



Presentation $(i = 1, \ldots, g,\ j = 1, \ldots, n)$

$$\Gamma = \left\{ a_1, b_i, c_j, \left| \prod_{i=1}^{g} [a_i, b_i] c_1 \cdots c_n = 1, c_j^{\nu_j} = 1 \right. \right\}$$

Good orbifolds $\Sigma = \underline{B}\Gamma = \Gamma \backslash \underline{E}\Gamma$

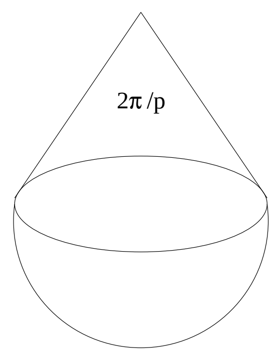
(orbifold-covered by a smooth manifold)

$$\Sigma_{g'} \xrightarrow{G} \Sigma(g; \nu_1, \ldots, \nu_n) = \Gamma \backslash \mathbb{H}^2$$

$\Sigma_{g'}$ = smooth compact Riemann surface genus

$$g' = 1 + \frac{\#G}{2}(2(g-1) + (n - \nu))$$

$\nu = \sum_{j=1}^{n} \nu_j^{-1}$



Exception: teardrop orbifold

Orbifold Euler characteristic $\chi_{orb}(\Sigma) \in \mathbb{Q}$

Multiplicative over orbifold covers, usual χ for smooth, inclusion–exclusion

$$\begin{aligned}\chi_{orb}(\Sigma_1 \cup \cdots \cup \Sigma_r) = & \sum_i \chi_{orb}(\Sigma_i) - \sum_{i,j} \chi_{orb}(\Sigma_i \cap \Sigma_j) \\ \cdots & +(-1)^{r+1} \chi_{orb}(\Sigma_1 \cap \cdots \cap \Sigma_r)\end{aligned}$$

$$\chi_{orb}(\Sigma(g; \nu_1, \ldots, \nu_n)) = 2 - 2g + \nu - n$$

Spectral theory

$$H_\sigma \in \mathbb{C}(\Gamma,\sigma) \subset C^*_r(\Gamma,\sigma) \subset \mathcal{U}(\Gamma,\sigma)$$

Spectral projections $P_E = \chi_{(-\infty,E]}(H_\sigma) \in \mathcal{U}(\Gamma,\sigma)$

$$E \notin \mathrm{Spec}(H_\sigma) \quad \Rightarrow \quad P_E \in C^*_r(\Gamma,\sigma)$$

$\exists F$ holom function on neighb of $\mathrm{Spec}(H_\sigma)$

$$P_E = \chi_{(-\infty,E]}(H_\sigma) = F(H_\sigma) = \int_C \frac{d\lambda}{\lambda - H_\sigma}$$

$C =$ contour around $\mathrm{Spec}(H_\sigma)$ left of E

Counting gaps in the spectrum $\Leftrightarrow$ counting projections in $C^*_r(\Gamma,\sigma)$

Trace $\quad \tau(T) = \langle Te_1, e_1\rangle_{\ell^2(\Gamma)}$

$$\mathrm{tr} = \tau \otimes \mathrm{Tr} : \mathrm{Proj}(C^*_r(\Gamma,\sigma)\otimes\mathcal{K}) \to \mathbb{R}$$

[tr] on $K_0(C^*_r(\Gamma,\sigma))$

Range of the Trace

Morita equiv $(A \otimes C_0(G)) \rtimes \Gamma \simeq C_0(\Gamma\backslash G, \mathcal{E})$
$\mathcal{E} = A \times_\Gamma G \to \Gamma\backslash G$

$$K_\bullet(C_r^*(\Gamma)) \cong K^\bullet_{SO(2)}(P(g;\nu_1,\ldots,\nu_n))$$

$$\cong K^\bullet_{orb}(\Sigma(g;\nu_1,\ldots,\nu_n)) \cong \begin{cases} \mathbb{Z}^{2-n+\sum \nu_j} & \bullet = even \\ \mathbb{Z}^{2g} & \bullet = odd \end{cases}$$

Twisted case $C_0(\Gamma\backslash G, \mathcal{E}) \simeq C_0(\Gamma\backslash G, \mathcal{E}_\sigma)$ provided

$$\delta(\sigma) = 0$$

$\delta : H^2(\Gamma, U(1)) \to H^3(\Gamma, \mathbb{Z})$ surjection from
long exact sequence of $1 \to \mathbb{Z} \hookrightarrow \mathbb{R} \stackrel{\exp(2\pi i\cdot)}{\longrightarrow} U(1) \to 1$

$$[\mathrm{tr}](K_0(C_r^*(\Gamma,\sigma))) = \mathbb{Z} + \theta\mathbb{Z} + \sum_j \nu_j^{-1}\mathbb{Z}$$

$\theta = \langle [\sigma], [\Gamma] \rangle$ rational $\Rightarrow$ finitely many gaps

Fundamental class $[\Gamma] = \frac{[\Sigma_{g'}]}{\#G}$

$\theta =$ irrational ?

Dense $*$-subalgebra

For $\gamma \in \Gamma$ set $De_\gamma = \ell(\gamma)e_\gamma$ with $\ell(\gamma) =$ word length

Unbounded closed derivation $\delta = [D, \cdot]$ on $C_r^*(\Gamma, \sigma)$

$$\mathcal{R} := \bigcap_{k \in \mathbb{N}} \mathrm{Dom}(\delta^k)$$

$\mathbb{C}(\Gamma, \sigma) \subset \mathcal{R}$, closed under holo functional calculus

Polynomial growth group cocycles on Γ define cyclic cocycles continuous on $\mathcal{R}$ (Haagerup type inequality)

$P_E = \chi_{(-\infty,E]}(H_\sigma + V) \in \mathcal{R}$

Cyclic cocycles $\quad t : \mathcal{R} \times \cdots \times \mathcal{R} \to \mathbb{C}$

$$\begin{aligned} t(a_0, a_1, \ldots, a_n) = & \; t(a_n, a_0, a_1, \ldots, a_{n-1}) \\ \ldots = & \; t(a_1, \ldots, a_n, a_0) \end{aligned}$$

and

$$\begin{aligned} t(aa_0, a_1, \ldots, a_n) & \; -t(a, a_0a_1, \ldots, a_n) \\ \cdots & \; (-1)^{n+1} t(a_n a, a_0, \ldots, a_{n-1}) = 0 \end{aligned}$$

Cyclic cocycles pair with K-theory

Conductance

In lattice $\Gamma = \mathbb{Z}^2$, current density in e_1 direction functional derivative δ_1 of H_σ by A_1 (component of magnetic potential)

Expected value of current $\mathrm{tr}(P\delta_1 H)$ for state P

Using $\partial_t P = i[P, H]$ and $\partial_t = \frac{\partial A_2}{\partial t} \times \delta_2$ (for $e_2 \perp e_1$) get

$$i\mathrm{tr}(P[\partial_t P, \delta_1 P]) = -iE_2 \mathrm{tr}(P[\delta_2 P, \delta_1 P])$$

(electrostatic potential gauged away: $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$)

In zero temperature limit charge carriers occupy all levels below Fermi level, so $P = P_F$

Conductance

$$\sigma_H = \mathrm{tr}(P_F[\delta_1 P_F, \delta_2 P_F])$$

In curved geometry

On Riemann surfaces changes of potential by real and imaginary parts of holomorphic 1-forms

$H^1(\Gamma, \mathbb{Z}) = \mathbb{Z}^{2g} \quad \{a_i, b_i\}_{i=1,\dots,g}$ symplectic basis

By effect of electron-electron interaction, to a moving elector the directions $\{e_1, e_2\}$ appear split into $\{e_i, e_{i+g}\}_{i=1,\dots,g}$ corresponding to a_i, b_i

Kubo formula

1-cocycle a on $\Gamma \Rightarrow$ derivation

$$\delta_a(f)(\gamma) = a(\gamma) f(\gamma) \quad \text{on} \quad f \in \mathbb{C}(\Gamma, \sigma)$$

Conductance cocycle:

$$\sum_{i=1}^{g} \operatorname{tr}\left(f^0 \left(\delta_{a_i}(f^1)\delta_{b_i}(f^2) - \delta_{b_i}(f^1)\delta_{a_i}(f^2)\right)\right)$$

defines cyclic 2-cocycle $\operatorname{tr}^K(f^0, f^1, f^2)$ on $\mathcal{R}$

Area cocycle On $G = \mathrm{PSL}(2,\mathbb{R})$ area cocycle

$$C(\gamma_1, \gamma_2) = \mathrm{Area}(\Delta(x_0, \gamma_1^{-1} x_0, \gamma_2 x_0)$$

hyperbolic area of geodesic triangle with given vertices

Restriction of $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ area cocycle on Γ

Area cocycle:

$$\sum_{\gamma_0\gamma_1\gamma_2=1} f^0(\gamma_0) f^1(\gamma_1) f^2(\gamma_2) C(\gamma_1, \gamma_2) \sigma(\gamma_1, \gamma_2)$$

defines cyclic 2-cocycle $\mathrm{tr}^C(f^0, f^1, f^2)$ on $\mathcal{R}$

Coboundaries t_1, t_2 cyclic 2-cocycles

$$\begin{aligned} t_1(a_0, a_1, a_2) - t_2(a_0, a_1, a_2) = & \lambda(a_0 a_1, a_2) - \lambda(a_0, a_1 a_2) \\ + & \lambda(a_2 a_0, a_1), \end{aligned}$$

where λ is a cyclic 1-cocycle

Conductance and area differ by coboundary

Comparison Difference between the hyperbolic area of a geodesic triangle and the Euclidean area of its image under the Abel-Jacobi map (curve and Jacobian)

$$U(\gamma_1, \gamma_2) = h(\gamma_2^{-1}, 1) - h(\gamma_1^{-1}, \gamma_2) + h(1, \gamma_1)$$

each term difference of line integrals, one along a geodesic segment in $\mathbb{H}^2$ and one along a straight line in the universal cover of the Jacobian

$$\mathrm{tr}_K(f_0, f_1, f_2) - \mathrm{tr}_C(f_0, f_1, f_2) =$$

$$\sum_{\gamma_0\gamma_1\gamma_2=1} f_0(\gamma_0) f_1(\gamma_1) f_2(\gamma_2) U(\gamma_1, \gamma_2) \sigma(\gamma_1, \gamma_2)$$

This can be written as $\lambda(f_0 f_1, f_2) - \lambda(f_0, f_1 f_2) + \lambda(f_2 f_0, f_1)$

$$\lambda(f_0, f_1) = \sum_{\gamma_0\gamma_1=1} f_0(\gamma_0) f_1(\gamma_1) h(1, \gamma_1) \sigma(\gamma_0, \gamma_1)$$

$\Rightarrow$ $[\mathrm{tr}^K] = [\mathrm{tr}^C]$ same values on K-theory

<u>Values of the conductance</u>

(Connes–Moscovici higher index theorem, twisted)

$$\mathrm{Ind}_{c,\Gamma,\sigma}(\not\partial_{\mathcal{E}}^{+} \otimes \nabla) = \frac{1}{2\pi \# G} \int_{\Sigma_{g'}} \hat{A}\ \mathrm{tr}(e^{R_{\mathcal{E}}}) e^{\omega} u_c$$

$\omega = d\eta$ 2-form of magnetic field $\nabla^2 = i\omega$

cocycle c and lift u_c to 2-form on $\Sigma_{g'}$

By dimension

$$\mathrm{Ind}_{c,\Gamma,\sigma}(\not\partial_{\mathcal{E}}^{+} \otimes \nabla) = \frac{\mathrm{rank}\mathcal{E}}{2\pi \# G} \int_{\Sigma_{g'}} u_c$$

Dependence on magnetic field only through $\mathcal{E}$ = orbifold bundle representing class of P_E spectral projection in $K_0(C_r^*(\Gamma, \sigma)$ (using Baum–Connes)

For area cocycle c 2-form u_c is hyperbolic volume form

$$\int_{\Sigma_{g'}} u_c = 2\pi(2g' - 2)$$

Orbifold Euler characteristic

$$\frac{(2g'-2)}{\# G} = -\chi_{orb}(\Sigma) \in \mathbb{Q}$$

Rational values of the onductance

$$\begin{aligned}\sigma_H = & \ \mathrm{tr}^K(P_F, P_F, P_F) \\ = & \ \mathrm{tr}^C(P_F, P_F, P_F) \in \mathbb{Z}\chi_{orb}(\Sigma)\end{aligned}$$

experimental	$g = 0$ $n = 3$
1/3	$\Sigma(0; 3, 6, 6)$
2/5	$\Sigma(0; 5, 5, 5)$
2/3	$\Sigma(0; 9, 9, 9)$
3/5	$\Sigma(0; 5, 10, 10)$
4/9	$\Sigma(0; 3, 9, 9)$
5/9	$\Sigma(0; 6, 6, 9)$
4/5	$\Sigma(0; 15, 15, 15)$
3/7	$\Sigma(0; 4, 4, 14)$
4/7	$\Sigma(0; 7, 7, 7)$
5/7	$\Sigma(0; 7, 14, 14)$

predicted	$g = 0$ $n = 3$
8/15	$\Sigma(0; 5, 6, 10)$
11/15	$\Sigma(0; 10, 10, 15)$
7/9	$\Sigma(0; 12, 12, 18)$
11/21	$\Sigma(0; 6, 6, 7)$
16/21	$\Sigma(0; 12, 12, 14)$

Problem: does not discriminate against even denominators (too many fractions)

Relation to Chern–Simons approach?

NC versions of Bloch varieties and periods?