Renormalization and the Riemann-Hilbert correspondence

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Commutative Hopf algebras and affine group scheme

$k =$ field of characteristic zero

$\mathcal{H}$ *commutative* algebra$/k$ with unit

coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H}$, counit $\varepsilon : \mathcal{H} \to k$, antipode $S : \mathcal{H} \to \mathcal{H}$

$$\begin{aligned} (\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \quad & : \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H} \otimes_k \mathcal{H}, \\ (id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta \quad & : \mathcal{H} \to \mathcal{H}, \\ m(id \otimes S)\Delta = m(S \otimes id)\Delta = 1\,\varepsilon \quad & : \mathcal{H} \to \mathcal{H}, \end{aligned}$$

Covariant functor G from $\mathcal{A}_k$ (commutative k-alg with 1) to $\mathcal{G}$ (groups)

$$G(A) = \mathrm{Hom}_{\mathcal{A}_k}(\mathcal{H}, A)$$

affine group scheme

Examples:

- Additive group $G = \mathbb{G}_a$: Hopf algebra $\mathcal{H} = k[t]$ with $\Delta(t) = t \otimes 1 + 1 \otimes t$.

- Multiplicative group $G = \mathbb{G}_m$: Hopf algebra $\mathcal{H} = k[t, t^{-1}]$ with $\Delta(t) = t \otimes t$.

- Roots of unity μ_n: Hopf algebra $\mathcal{H} = k[t]/(t^n - 1)$.

- $G = \mathrm{GL}_n$: Hopf algebra

$$\mathcal{H} = k[x_{i,j}, t]_{i,j=1,\ldots,n} / \det(x_{i,j})t - 1,$$

  with $\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}$.

- $\mathcal{H}$ fin. gen. alg./k: $G \subset \mathrm{GL}_n$ linear algebraic group/k.

- $\mathcal{H} = \cup_i \mathcal{H}_i$, $\Delta(\mathcal{H}_i) \subset \mathcal{H}_i \otimes \mathcal{H}_i$, $S(\mathcal{H}_i) \subset \mathcal{H}_i$: projective limit of linear algebraic groups

$$G = \varprojlim_i G_i$$

Lie algebra: functor $\mathfrak{g}$ from $\mathcal{A}_k$ to Lie

$\mathfrak{g}(A) = \{L : \mathcal{H} \to A \,|\, L(X\,Y) = L(X)\,\varepsilon(Y) + \varepsilon(X)\,L(Y)\}$

Milnor-Moore: $\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n$, with $\mathcal{H}_0 = k$ and $\mathcal{H}_n$ fin dim/k. Dual $\mathcal{H}^\vee$ with primitive elements $\mathcal{L}$:

$$\mathcal{H} = U(\mathcal{L})^\vee$$

Reconstruct $\mathcal{H}$ from the Lie algebra $\mathcal{L} = \mathfrak{g}(k)$.

For $\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n$ action of $\mathbb{G}_m$

$$u^Y(X) = u^n\,X, \ \forall X \in \mathcal{H}_n, \, u \in \mathbb{G}_m$$

$G^* = G \rtimes \mathbb{G}_m$

Connes–Kreimer theory

Perturbative QFT

$\mathcal{T}$ = scalar field theory in dimension D

$$S(\phi) = \int \mathcal{L}(\phi)\, d^D x = S_0(\phi) + S_{\text{int}}(\phi)$$

with Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\,\phi^2 - \mathcal{L}_{\text{int}}(\phi)$$

Effective action (perturbative expansion):

$$S_{eff}(\phi) = S_0(\phi) + \sum_{\Gamma \in \text{1PI}} \frac{\Gamma(\phi)}{\#\text{Aut}(\Gamma)}$$

$$\Gamma(\phi) = \frac{1}{N!}\int_{\sum p_j = 0} \widehat{\phi}(p_1)\ldots\widehat{\phi}(p_N)\, U^z_\mu(\Gamma(p_1,\ldots,p_N))\, dp_1\ldots dp_N$$

$$U(\Gamma(p_1,\ldots,p_N)) = \int d^D k_1 \cdots d^D k_L \; I_\Gamma(k_1,\cdots k_L, p_1, \cdots p_N)$$

$U^z_\mu(\Gamma(p_1,\ldots,p_N))$: DimReg+MS

$$= \int \mu^{zL}\, d^{D-z} k_1 \cdots d^{D-z} k_L \; I_\Gamma(k_1,\cdots k_L, p_1, \cdots p_N)$$

Laurent series in z

BPHZ renormalization scheme

Class of subgraphs $\mathcal{V}(\Gamma)$:

$\mathcal{T}$ renormalizable theory, $\Gamma =$ 1PI Feynman graph: $\mathcal{V}(\Gamma)$ (not necessarily connected) subgraphs $\gamma \subset \Gamma$ with

1. Edges of γ are internal edges of Γ.
2. Let $\tilde{\gamma}$ be a graph obtained by adjoining to a connected component of γ the edges of Γ that meet the component. Then $\tilde{\gamma}$ is a Feynman graph of the theory $\mathcal{T}$.
3. The unrenormalized value $U(\tilde{\gamma})$ is divergent.
4. The graph Γ/γ is a Feynman graph of the theory.
5. The components of γ are 1PI graphs.
6. The graph Γ/γ is a 1PI graph.

BPHZ procedure:

Preparation:

$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

Coefficient of the pole part is given by a *local term*

Counterterms:

$$C(\Gamma) = -T(\overline{R}(\Gamma))$$

$$= -T\left(U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)\right)$$

$T =$ projection on the polar part of the Laurent series

Renormalized value:

$$R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma)$$

$$= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

Connes-Kreimer Hopf algebra of Feynman graphs

Discrete version (over $k = \mathbb{C}$, in fact $k = \mathbb{Q}$)

$\mathcal{H} = \mathcal{H}(\mathcal{T})$ depends on the theory $\mathcal{T}$

Generators: 1PI graphs Γ of the theory

Grading: $\deg(\Gamma_1 \cdots \Gamma_r) = \sum_i \deg(\Gamma_i)$
and $\deg(1) = 0$

Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

Antipode: inductively (lower deg)

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Affine group scheme $G(\mathcal{H}(\mathcal{T})) = \mathrm{Difg}(\mathcal{T})$
"diffeographisms"

$$\mathrm{Difg}(\mathcal{T}) \to \mathrm{Diff}$$

to formal diffeomorphisms of the coupling constants

$$g_{\mathrm{eff}} = g + \sum_n \alpha_n g^n, \quad \alpha_n \in \mathcal{H}$$

Lie algebra: (Milnor-Moore)

$$[\Gamma, \Gamma'] = \sum_v \Gamma \circ_v \Gamma' - \sum_{v'} \Gamma' \circ_{v'} \Gamma$$

$\Gamma \circ_v \Gamma'$ = inserting Γ' in Γ at the vertes v

<u>Continuous version</u> On $E_\Gamma := \{(p_i)_{i=1,\dots,N} \; ; \; \sum p_i = 0\}$ distributions

$$C_c^{-\infty}(E) = \oplus_\Gamma C_c^{-\infty}(E_\Gamma)$$

Hopf algebra

$$\tilde{\mathcal{H}}(\mathcal{T}) = \mathrm{Sym}(C_c^{-\infty}(E))$$

$$\Delta(\Gamma, \sigma) = (\Gamma, \sigma) \otimes 1 + 1 \otimes (\Gamma, \sigma) + \sum_{\gamma \in \mathcal{V}(\mathcal{T});\, i \in \{0,1\}} (\gamma_{(i)}, \sigma_i) \otimes (\Gamma / \gamma_{(i)}, \sigma)$$

Loops and Birkhoff factorization

$\Delta =$ (infinitesimal) disk around $z = 0$, $C = \partial\Delta$

$C_+ \cup C_- = \mathbb{P}^1(\mathbb{C}) \smallsetminus C$

$G(\mathbb{C}) =$ complex connected Lie group

loop $\gamma : C \to G(\mathbb{C})$

Birkhoff factorization problem: is it possible to factor

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z)$$

$\forall z \in C$, with $\gamma_\pm : C_\pm \to G(\mathbb{C})$ holomorphic, $\gamma_-(\infty) = 1$

In general <u>no</u>: for $G(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$ only

$$\gamma(z) = \gamma_-(z)^{-1}\,\lambda(z)\,\gamma_+(z)$$

$\lambda(z)$ diagonal $(z^{k_1}, z^{k_2}, \ldots, z^{k_n})$: nontrivial holomorphic vector bundles on $\mathbb{P}^1(\mathbb{C})$ with $c_1(L_i) = k_i$ and

$$E = L_1 \oplus \ldots \oplus L_n$$

$\mathcal{H}$ commutative Hopf algebra over $\mathbb{C}$:
$K = \mathbb{C}(\{z\}) = \mathbb{C}\{z\}[z^{-1}]$, $\mathcal{O} = \mathbb{C}\{z\}$, $\mathcal{Q} = z^{-1}\mathbb{C}[z^{-1}]$,
$\tilde{\mathcal{Q}} = \mathbb{C}[z^{-1}]$

loop $\gamma(z)$: element $\phi \in G(K) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, K)$

positive part $\gamma_+(z)$: element $\phi_+ \in G(\mathcal{O})$

negative part $\gamma_-(z)$: element $\phi_- \in G(\tilde{\mathcal{Q}})$
$\gamma_-(\infty) = 1 \Leftrightarrow \varepsilon_- \circ \phi_- = \varepsilon$

Birkhoff $\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z)$ becomes

$$\phi = (\phi_- \circ S) * \phi_+$$

Product $\phi_1 * \phi_2$ dual to coproduct

$$\langle \phi_1 * \phi_2, X \rangle = \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$$

$G =$ pro-unipotent affine group scheme of a commutative Hopf algebra $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$

Always have Birkhoff factorization: inductive formula (CK)

$$\phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\phi(X'')\right)$$

$$\phi_+(X) = \phi(X) + \phi_-(X) + \sum \phi_-(X')\phi(X'')$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

<u>BPHZ = Birkhoff</u> Take $G = \widetilde{\mathrm{Difg}}(\mathcal{T})$ (continuous version)

Data $U^z(\Gamma(p_1,\ldots,p_N))$: homomorphism $U : \tilde{\mathcal{H}}(\mathcal{T}) \to K$

$$(\Gamma,\sigma) \mapsto h(z) = \langle \sigma, U^z(\Gamma(p_1,\ldots,p_N))\rangle$$

Laurent series

$\phi = U$, $\phi_- = C$, $\phi_+ = R$: same as BPHZ!

Dependence on mass scale: $\gamma_\mu(z)$

$$\gamma_\mu(z) = \gamma_{\mu^-}(z)^{-1}\gamma_{\mu^+}(z)$$

Grading by loop number:
$Y(X) = n\,X$, $\forall X \in \mathcal{H}_n^\vee(\mathcal{T})$

$$\theta_t \in \mathrm{Aut}(\mathrm{Difg}(\mathcal{T})), \qquad \frac{d}{dt}\theta_t\,|_{t=0} = Y$$

Main properties of scale dependence:

$$(*) = \begin{cases} \gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)) \\ \frac{\partial}{\partial\mu}\,\gamma_{\mu^-}(z) = 0. \end{cases}$$

Renormalization group:

$$F_t = \lim_{z\to 0}\gamma_-(z)\,\theta_{tz}(\gamma_-(z)^{-1})$$

action $\gamma_{e^t\mu^+}(0) = F_t\,\gamma_{\mu^+}(0)$

Beta function: $\beta = \frac{d}{dt}F_t|_{t=0} \in \mathfrak{g}$

$$\beta := Y\,\mathrm{Res}\,\gamma, \qquad \mathrm{Res}_{z=0}\,\gamma := -\left(\frac{\partial}{\partial u}\,\gamma_-\left(\frac{1}{u}\right)\right)_{u=0}$$

Connes-Kreimer theory in a nutshell:

$G =$ pro-unipotent affine group scheme ($=$ Difg($\mathcal{T}$))

$L(G(\mathbb{C}), \mu) =$ loops $\gamma_\mu(z)$ with (*) properties

Divergences (counterterms) $\gamma_-(z)$

Renormalized values $\gamma_{\mu^+}(0)$

$\Rightarrow$ Understand data $L(G(\mathbb{C}), \mu)$ and $\gamma_-(z)$

Renormalization and the Riemann-Hilbert correspondence (AC–MM)

Tannakian formalism

Abelian category $\mathcal{C}$:

- $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ abelian groups
  ($\exists 0 \in Obj(\mathcal{C})$ with $\mathrm{Hom}_{\mathcal{C}}(0,0)$ trivial group)

- There are products and coproducts: $\forall X, X' \in Obj(\mathcal{C})$, $\exists Y \in Obj(\mathcal{C})$ and

  $$X \xrightarrow{f_1} Y \xleftarrow{f_2} X' \quad \text{and} \quad X \xleftarrow{h_1} Y \xrightarrow{h_2} X',$$

  with $h_1 f_1 = 1_X$, $h_2 f_2 = 1_{X'}$, $h_2 f_1 = 0 = h_1 f_2$, $f_1 h_2 + f_2 h_1 = 1_Y$.

- There are Kernels and Cokernels: $\forall X, Y \in Obj(\mathcal{C})$, $\forall f : X \to Y$ can decompose $j \circ i = f$,

  $$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K',$$

  with $K = \mathrm{Ker}(f)$, $K' = \mathrm{Coker}(f)$, and $I = \mathrm{Ker}(k) = \mathrm{Coker}(c)$.

k-linear category $\mathcal{C}$: $\text{Hom}_{\mathcal{C}}(X,Y)$ is a k-vector space $\forall X,Y \in Obj(\mathcal{C})$.

Tensor category $\mathcal{C}$: k-linear with $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

- $\exists 1 \in Obj(\mathcal{C})$ with $\text{End}(1) \cong k$ and functorial isomorphisms

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

$$l_X : X \otimes 1 \to X \quad \text{and} \quad r_X : 1 \otimes X \to X.$$

- Commutativity: $c_{Y,X} = c_{X,Y}^{-1}$

Rigid tensor category $\mathcal{C}$: tensor with duality $\vee : \mathcal{C} \to \mathcal{C}^{op}$

- $\forall X \in Obj(\mathcal{C})$ the functor $- \otimes X^\vee$ is left adjoint to $- \otimes X$ and the functor $X^\vee \otimes -$ is right adjoint to $X \otimes -$.

- Evaluation morphism $\epsilon : X \otimes X^\vee \to 1$ and unit morphism $\delta : 1 \to X^\vee \otimes X$ with $(\epsilon \otimes 1) \circ (1 \otimes \delta) = 1_X$ and $(1 \otimes \epsilon) \circ (\delta \otimes 1) = 1_{X^\vee}$.

Functors $\omega : \mathcal{C} \to \mathcal{C}'$

faithful: $\omega : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}'}(\omega(X), \omega(Y))$ injection

additive: $\omega : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}'}(\omega(X), \omega(Y))$ k-linear

exact: $0 \to X \to Y \to Z \to 0$ exact $\Rightarrow$ $0 \to \omega(X) \to \omega(Y) \to \omega(Z) \to 0$ exact

tensor: functorial isomorphisms $\tau_1 : \omega(1) \to 1$ and $\tau_{X,Y} : \omega(X \otimes Y) \to \omega(X) \otimes \omega(Y)$

Fiber functor, Tannakian categories $\mathcal{C}$ be a k-linear rigid tensor category: fiber functor $\omega : \mathcal{C} \to \text{Vect}_K$ exact faithful tensor functor, K extension of k.

$\Rightarrow$ $\mathcal{C}$ Tannakian (=has fiber functor), neutral Tannakian ($K = k$)

(Grothendieck, Savendra-Rivano, Deligne, . . .)
$\mathcal{C}$ neutral Tannakian $\Rightarrow$ $\mathcal{C} \cong \text{Rep}_G$
$G = \underline{\text{Aut}}^{\otimes}(\omega)$ affine group scheme Gal($\mathcal{C}$)

Example: $\mathrm{Rep}_{\mathbb{Z}} \cong \mathrm{Rep}_G$ affine group scheme $G = \bar{\mathbb{Z}}$ dual to $\mathcal{H} = \mathbb{C}[e(q), t]$, for $q \in \mathbb{C}/\mathbb{Z}$, with relations $e(q_1 + q_2) = e(q_1)e(q_2)$ and coproduct $\Delta(e(q)) = e(q) \otimes e(q)$ and $\Delta(t) = t \otimes 1 + 1 \otimes t$.

Riemann–Hilbert correspondence

Tannakian formalism applied to categories of differential systems (differential Galois theory)

$(K, \delta) =$ differential field
e.g. $K = \mathbb{C}\{z\}[z^{-1}]$ or $K = \mathbb{C}((z))$

Category $\mathcal{D}_K$ of differential modules over K:
Objects (V, ∇), vector space $V \in Obj(\mathcal{V}_K)$ and connection
$\mathbb{C}$-linear map $\nabla : V \to V$ with $\nabla(fv) = \delta(f)v + f\nabla(v)$, for all $f \in K$ and all $v \in V$
Morphisms $\mathrm{Hom}((V_1, \nabla_1), (V_2, \nabla_2))$ K-linear maps $T : V_1 \to V_2$ with $\nabla_2 \circ T = T \circ \nabla_1$

$(V_1, \nabla_1) \otimes (V_2, \nabla_2) = (V_1 \otimes V_2, \nabla_1 \otimes 1 + 1 \otimes \nabla_2)$
and dual $(V, \nabla)^\vee$

Fiber functor $\omega(V,\nabla) = \mathrm{Ker}\nabla$. Neutral Tannakian category $\mathcal{D}_K \cong \mathrm{Rep}_G$

For $K = \mathbb{C}((z))$, affine group scheme $G = \mathcal{T} \rtimes \bar{\mathbb{Z}}$ of Ramis exponential torus $\mathcal{T} = \mathrm{Hom}(\mathcal{B}, \mathbb{C}^*)$ with $\mathcal{B} = \cup_{\nu\in\mathbb{N}} \mathcal{B}\nu$, for $\mathcal{B}_\nu = z^{-1/\nu}\mathbb{C}[z^{-1/\nu}]$.

For $K = \mathbb{C}\{z\}[z^{-1}]$ extra generators: Stokes phenomena (resummation of divergent series in sectors)

Example: ODE $\delta(u) = Au$, subcategory of $\mathcal{D}_K \Rightarrow$ differential Galois group (Aut of Picard-Vessiot ring)

Example: ODE $\delta(u) = Au$ regular-singular iff $\exists T$ invertible matrix coeff. in $K = \mathbb{C}((z))$, with $T^{-1}AT - T^{-1}\delta(T) = B/z$, B coeff. in $\mathbb{C}[[z]]$. Tannakian subcategory $\mathcal{D}_K^{rs}$ of $\mathcal{D}_K$ gen. by regular-singular equations $\mathcal{D}_K^{rs} \cong \mathrm{Rep}_{\bar{\mathbb{Z}}}$ (monodromy $\mathbb{Z} = \pi_1(\Delta^*)$)

Claim: There is a Riemann-Hilbert correspondence associated to the data of perturbative renormalization

- Not just over the disk Δ but a $\mathbb{C}^*$-fibration B over Δ, so we exit from the category $\mathcal{D}_K$.

- Equivalence relation on connections by gauge transformations regular at $z = 0$.

- Class of connections (equisingular connections) not regular-singular: setting of “irregular” Riemann–Hilbert correspondence with arbitrary degree of irregularity, as for $\mathcal{D}_K$.

- The Galois group same in formal and non-formal case (no Stokes phenomena).

Data of CK revisited

G = pro-unipotent affine group scheme (= Difg($\mathcal{T}$))
$L(G(\mathbb{C}),\mu)$ = loops $\gamma_\mu(z)$ with

$$(*) = \begin{cases} \gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)) \\ \frac{\partial}{\partial\mu}\gamma_{\mu^-}(z) = 0. \end{cases}$$

Divergences (counterterms) $\gamma_-(z)$

First step (CK):

$$\gamma_-(z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n}$$

coefficients $d_n \in \mathcal{H}^\vee$

$$Y\, d_{n+1} = d_n\, \beta \quad \forall n \geq 1, \quad \text{and} \quad Y d_1 = \beta$$

$\Rightarrow$ Can write as iterated integrals

Time ordered exponential

$\mathfrak{g}(\mathbb{C})$-valued smooth $\alpha(t)$, $t \in [a,b] \subset \mathbb{R}$

$$\mathsf{T}e^{\int_a^b \alpha(t)\,dt} := 1 + \sum_1^\infty \int_{a \le s_1 \le \cdots \le s_n \le b} \alpha(s_1) \cdots \alpha(s_n)\; ds_1 \cdots ds_n$$

product in $\mathcal{H}^\vee$, with $1 \in \mathcal{H}^\vee$ counit ε of $\mathcal{H}$

- Paired with $X \in \mathcal{H}$ the sum is finite.
- Defines an element of $G(\mathbb{C})$.
- Value $g(b)$ of unique solution $g(t) \in G(\mathbb{C})$ with $g(a) = 1$ of

$$dg(t) = g(t)\,\alpha(t)\,dt$$

- Multiplicative over sum of paths:

$$\mathsf{T}e^{\int_a^c \alpha(t)\,dt} = \mathsf{T}e^{\int_a^b \alpha(t)\,dt}\,\mathsf{T}e^{\int_b^c \alpha(t)\,dt}$$

- $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$, then

$$\gamma_-(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\, dt}$$

by $\gamma_-(z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n}$ with

$$d_n = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta)\, \theta_{-s_2}(\beta) \ldots \theta_{-s_n}(\beta)\; ds_1 \cdots ds_n$$

- $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$, then

$$\gamma_\mu(z) = \mathrm{T}e^{-\frac{1}{z}\int_\infty^{-z\log\mu} \theta_{-t}(\beta)\, dt}\; \theta_{z\log\mu}(\gamma_{\mathrm{reg}}(z))$$

for a unique $\beta \in \mathfrak{g}(\mathbb{C})$ (with $\gamma_{\mathrm{reg}}(z)$ a loop regular at $z = 0$)

The Birkhoff factorization

$$\gamma_{\mu^+}(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^{-z\log\mu} \theta_{-t}(\beta)\, dt}\, \theta_{z\log\mu}(\gamma_{\mathrm{reg}}(z))$$

$$\gamma_-(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\, dt}$$

Conversely, given $\beta \in \mathfrak{g}(\mathbb{C})$ and $\gamma_{\mathrm{reg}}(z)$ regular
$\Rightarrow \gamma_\mu \in L(G(\mathbb{C}), \mu)$

$\varpi = \alpha(s,t)ds + \eta(s,t)dt$ flat $\mathfrak{g}(\mathbb{C})$-valued connection

$$\partial_s \eta - \partial_t \alpha + [\alpha, \eta] = 0$$

$\mathsf{T}e^{\int_0^1 \gamma^* \varpi}$ depends on homotopy class of path

Differential field (K, δ) with $\mathrm{Ker}\delta = \mathbb{C}$
log derivative on $G(K)$

$$D(f) := f^{-1} f' \in \mathfrak{g}(K)$$

$f'(X) = \delta(f(X)), \quad \forall X \in \mathcal{H}$

Differential equation $D(f) = \varpi$

Existence of solutions: trivial monodromy

$G = \varprojlim_i G_i$, monodromy

$$M_i(\varpi)(\gamma) := \mathsf{T}e^{\int_0^1 \gamma^* \varpi}$$

punctured disk Δ_i^* of positive radius

$$M(\varpi) = 1$$

well defined on G

(K,δ), $d : K \to \Omega^1$, $df = \delta(f)\,dz$

$$D : G(K) \to \Omega^1(\mathfrak{g}), \qquad Df = f^{-1}\,df$$

$$D(fh) = Dh + h^{-1}\,Df\,h$$

Two connections ϖ and ϖ' are equivalent iff

$$\varpi' = Dh + h^{-1}\varpi\,h, \qquad \text{with} \quad h \in G(\mathcal{O})$$

Equivalent $\Leftrightarrow$ same negative part of Birkhoff: $D(f^{\varpi}) = \varpi$ and $D(f^{\varpi'}) = \varpi'$ solutions in $G(K)$

$$\varpi \sim \varpi' \Longleftrightarrow f_-^{\varpi} = f_-^{\varpi'}$$

Flat equisingular connections: accounts for μ-dependence

Principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle $\mathbb{G}_m \to B \xrightarrow{\pi} \Delta$ over infinitesimal disk Δ.

$P = B \times G$, $P^* = P|_{B^*}$, $B^* = B|_{\Delta^*}$

Action of $\mathbb{G}_m$ by $b \mapsto u(b)$, $\forall u \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ and action of $\mathbb{G}_m$ on G dual to graded Hopf algebra $\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n$

$$u(b, g) = (u(b), u^Y(g)), \qquad \forall u \in \mathbb{G}_m$$

Flat connection ϖ on P^* is *equisingular* iff

- ϖ is $\mathbb{G}_m$-invariant

$$\varpi(z, u(v)) = u^Y(\varpi(z, v)), \quad \forall u \in \mathbb{G}_m$$

$v = (\sigma(z), g)$, for $z \in \Delta$ and $g \in G$

- all the restrictions are equivalent

$$\sigma_1^*(\varpi) \sim \sigma_2^*(\varpi)$$

σ_1 and σ_2 are two sections of B as above, with $\sigma_1(0) = y_0 = \sigma_2(0)$

The connections $\sigma_1^*(\varpi)$ and $\sigma_2^*(\varpi)$ have the same type of singularity at the origin $z = 0$

Equivalence: ϖ and ϖ' on P^* equivalent iff

$$\varpi' = Dh + h^{-1}\varpi h,$$

with h a G-valued $\mathbb{G}_m$-invariant map regular in B.

Thm: Bijective correspondence between equivalence classes of flat equisingular G-connections ϖ on P^* and elements $\beta \in \mathfrak{g}(\mathbb{C})$
$\varpi \sim D\gamma$ with

$$\gamma(z,v) = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}$$

(integral on the path $u = tv$, $t \in [0,1]$)
Correspondence independent of choice of section $\sigma : \Delta \to B$ with $\sigma(0) = y_0$.

Key step: vanishing of monodromies around Δ^* and $\mathbb{C}^*$

Category of equivariant flat vector bundles

$V = \oplus_{n\in\mathbb{Z}} V_n$ fin dim $\mathbb{Z}$-graded vector space; trivial vector bundle $E = B \times V$ filtered by

$$W^{-n}(V) = \oplus_{m\geq n} V_m$$

$\mathbb{G}_m$ action induced by grading.

W-connection on a filtered vector bundle (E, W) over B:

$$W^{-n-1}(E) \subset W^{-n}(E),$$

$$Gr_n^W(E) = W^{-n}(E)/W^{-n-1}(E)$$

Connection ∇ on $E^* = E|_{B^*}$, compatible with filtration: restricts to $W^{-n}(E^*)$ and induces trivial connection on $Gr^W(E)$

Two W-connections ∇_i on E^* are W-equivalent iff $\exists T \in$ Aut(E), preserving filtration, inducing identity on $Gr^W(E)$, with $T \circ \nabla_1 = \nabla_2 \circ T$

A W-connection ∇ on E is equisingular if it is $\mathbb{G}_m$-invariant and all restrictions to sections $\sigma : \Delta \to B$ with $\sigma(0) = y_0$ are W-equivalent.

Category $\mathcal{E}$ equisingular flat vector bundles

$Obj(\mathcal{E})$ pairs $\Theta = (V, [\nabla])$

$V =$ fin dim $\mathbb{Z}$-graded vector space, $[\nabla] = W$-equivalence class of flat equisingular W-connection ∇ on $E^* = B^* \times V$

Morphisms: $T \in \mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$ linear map $T : V \to V'$
compatible with the grading and on $(E' \oplus E)^*$

$$\nabla_1 = \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix}$$

$$\nabla_2 = \begin{pmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{pmatrix}$$

are W-equivalent on B

(Notice: category of filtered vector spaces, with morphisms linear maps respecting filtration, is not an abelian category)

For $G = \text{Difg}(\mathcal{T})$, $\varpi =$ flat equisingular connection on $P^* = B^* \times G$, fin dim lin rep $\xi : G \to \text{GL}(V) \Rightarrow \Theta \in Obj(\mathcal{E})$. Equivalent ϖ give same Θ.

<u>THM</u> The category $\mathcal{E}$ is a neutral Tannakian category (over $\mathbb{C}$, over $\mathbb{Q}$)
with fiber functor $\omega(\Theta) = V$

$$\mathcal{E} \cong \text{Rep}_{\mathbb{U}^*}$$

$\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$ affine group scheme, $\mathbb{U} =$ prounipotent dual to Hopf algebra

$$\mathcal{H}_{\mathbb{U}} = U(\mathcal{L}_{\mathbb{U}})^{\vee}$$

$\mathcal{L}_{\mathbb{U}} = \mathcal{F}(1, 2, 3, \cdots)_{\bullet}$ denote the free graded Lie algebra generated by elements e_{-n} of degree n, for each $n > 0$

Renormalization group

$$e = \sum_{1}^{\infty} e_{-n}$$

determines $\mathbf{rg}\ : \mathbb{G}_a \to \mathbb{U}$

Universal singular frame

$$\gamma_{\mathbb{U}}(z,v) = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(e)\,\frac{du}{u}}$$

Universal source of counterterms

Coefficients:

$$\gamma_{\mathbb{U}}(z,v) = \sum_{n\geq 0}\sum_{k_j>0} \frac{e_{-k_1}e_{-k_2}\cdots e_{-k_n}}{k_1\,(k_1+k_2)\cdots(k_1+k_2+\cdots+k_n)}\, v^{\sum k_j}\, z^{-n}$$

(local index formula Connes-Moscovici)

Key step in proof of THM: for $\Theta = [V, \nabla]$ be an object of $\mathcal{E}$, there exists a unique representation $\rho = \rho_\Theta$ of $\mathbb{U}^*$ in V, such that

$$D\rho(\gamma_{\mathbb{U}}) \simeq \nabla$$

universal singular frame $\gamma_{\mathbb{U}}$

Note: $\mathbb{Q}(n) \in Obj(\mathcal{E})$ with V 1-dim over $\mathbb{Q}$ in deg n, ∇ trivial connection on assoc bundle E over B. Fiber functor:

$$\omega_n(\Theta) = \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}^W_{-n}(\Theta))$$

For $G = \text{Difg}(\mathcal{T})$, canonical bijection: equivalence classes of flat equisingular connections on P^* and graded representations

$$\rho : \mathbb{U}^* \to G^* = G \rtimes \mathbb{G}_m$$

Using the beta function:

$$\beta = \sum_{1}^{\infty} \beta_n$$

$Y(\beta_n) = n\beta_n$, representation $\mathbb{U} \to G$ compatible with $\mathbb{G}_m$:

$$e_{-n} \mapsto \beta_n$$

Action on physical constants through Difg $\to$ Diff map:

$$\mathbb{U} \to \text{Difg}(\mathcal{T}) \to \text{Diff}$$

Motives

Cohomologies for alg varieties:

de Rham $H^{\cdot}_{dR}(X) = \mathbb{H}^{\cdot}(X, \Omega^{\cdot}_X)$
Betti $H^{\cdot}_B(X, \mathbb{Q})$ (singular homology)
étale $H^i_{et}(\bar{X}, \mathbb{Q}_\ell)$ for $\ell \neq \mathrm{char}\, k$ and $\bar{X}$ over $\bar{k}$.

Isomorphisms: period isomorphism

$$H^i_{dR}(X, k) \otimes_\sigma \mathbb{C} \cong H^i_B(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

and comparison isom

$$H^i_B(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^i_{et}(\bar{X}, \mathbb{Q}_\ell)$$

Universal cohomology theory? Motives

Linearization of the category of algebraic varieties (adding morphisms; analog with Morita theory for algebras)

$$X \mapsto h(X) = \oplus_i h^i(X)$$

if $h^j = 0$, $\forall j \neq i$, pure of weight i
Pure motives "direct summands of algebraic varieties"

Pure Motives

Objects (X, p), $p = p^2 \in$ End(X), X smooth projective

Morphisms Hom(X, Y) *correspondences*: alg cycles in $X \times Y$, codim=dim X. Equivalences (numerical, rational,...)
Hom$((X, p), (Y, q)) = q$Hom$(X, Y)p$

Tate motives $\mathbb{Q}(1)$ inverse of $h^2(\mathbb{P}^1)$, $\mathbb{Q}(0) = h(pt)$, $\mathbb{Q}(n + m) = \mathbb{Q}(n) \otimes \mathbb{Q}(m)$

(Grothendieck standard conjectures)
Jannsen: numerical equivalence $\Rightarrow$ neutral Tannakian category (fiber functor Betti cohomology) $\Rightarrow$ Rep_G affine group scheme G

Tate motives $G = \mathbb{G}_m$.

Mixed motives

Extend "universal cohomology theory" to X not smooth projective: technically much more complicated, via constructions of derived category (Voevodsky, Levine, Hanamura)

Mixed Tate motives

(filtered: graded pieces Tate motives)

Full subcategory of Tate motives (over a field k or a scheme S) $\mathcal{MT}_{mix}(S)$ (Deligne–Goncharov)

Motivic Galois group of $\mathcal{MT}_{mix}(k)$ extension $G \rtimes \mathbb{G}_m$, G pro-unipotent, Lie (G) free one generator in each odd degree $n \leq -3$

THM(CM) (non-canonical) isomorphism $U^* \sim G_{\mathcal{M}_T}(\mathcal{O})$ with motivic Galois group of the scheme S_4 of 4-cyclotomic integers

$\mathcal{O} = \mathbb{Z}[i][1/2]$