Renormalization and the Riemann-Hilbert correspondence

Alain Connes and Matilde Marcolli

Santa Barbara, November 2005
Commutative Hopf algebras and affine group schemes

$k = \text{field of characteristic zero}$

$\mathcal{H}$ \textit{commutative} algebra$/k$ with unit

coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H}$, counit $\varepsilon : \mathcal{H} \to k$, antipode $S : \mathcal{H} \to \mathcal{H}$

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta : \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H} \otimes_k \mathcal{H},$$

$$(id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta : \mathcal{H} \to \mathcal{H},$$

$$m(id \otimes S)\Delta = m(S \otimes id)\Delta = 1 \varepsilon : \mathcal{H} \to \mathcal{H},$$

Covariant functor $G$ from $A_k$ (commutative $k$-alg with 1) to $G$ (groups)

$$G(A) = \text{Hom}_{A_k}(\mathcal{H}, A)$$

\textit{affine group scheme}
Examples:

• Additive group $G = \mathbb{G}_a$: Hopf algebra $\mathcal{H} = k[t]$ with $\Delta(t) = t \otimes 1 + 1 \otimes t$.

• Multiplicative group $G = \mathbb{G}_m$: Hopf algebra $\mathcal{H} = k[t, t^{-1}]$ with $\Delta(t) = t \otimes t$.

• Roots of unity $\mu_n$: Hopf algebra $\mathcal{H} = k[t]/(t^n - 1)$.

• $G = \text{GL}_n$: Hopf algebra

$$\mathcal{H} = k[x_{i,j}, t]_{i,j=1,...,n}/\det(x_{i,j})t - 1,$$

with $\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}$.

• $\mathcal{H}$ fin. gen. alg./$k$: $G \subset \text{GL}_n$ linear algebraic group/$k$.

• $\mathcal{H} = \bigcup_i \mathcal{H}_i$, $\Delta(\mathcal{H}_i) \subset \mathcal{H}_i \otimes \mathcal{H}_i$, $S(\mathcal{H}_i) \subset \mathcal{H}_i$: projective limit of linear algebraic groups

$G = \lim_{\leftarrow} G_i$
Lie algebra: functor $g$ from $A_k$ to $\text{Lie}$

\[ g(A) = \{ L : \mathcal{H} \rightarrow A \mid L(XY) = L(X)\varepsilon(Y) + \varepsilon(X)L(Y) \} \]

Milnor-Moore: $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$, with $\mathcal{H}_0 = k$ and $\mathcal{H}_n$ fin dim $/k$. Dual $\mathcal{H}^\vee$ with primitive elements $\mathcal{L}$:

\[ \mathcal{H} = U(\mathcal{L})^\vee \]

Reconstruct $\mathcal{H}$ from the Lie algebra $\mathcal{L} = g(k)$.

For $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ action of $\mathbb{G}_m$

\[ u^Y(X) = u^n X, \ \forall X \in \mathcal{H}_n, \ u \in \mathbb{G}_m \]

$G^* = G \rtimes \mathbb{G}_m$
Connes–Kreimer theory

**Perturbative QFT**

\[ \mathcal{T} = \text{scalar field theory in dimension } D \]

\[ S(\phi) = \int \mathcal{L}(\phi) \, d^D x = S_0(\phi) + S_{\text{int}}(\phi) \]

with Lagrangian density

\[ \mathcal{L}(\phi) = \frac{1}{2} (\partial\phi)^2 - \frac{m^2}{2} \phi^2 - \mathcal{L}_{\text{int}}(\phi) \]

Effective action (perturbative expansion):

\[ S_{\text{eff}}(\phi) = S_0(\phi) + \sum_{\Gamma \in \text{1PI}} \frac{\Gamma(\phi)}{\# \text{Aut}(\Gamma)} \]

\[ \Gamma(\phi) = \frac{1}{N!} \int \sum_{p_j=0} \hat{\phi}(p_1) \ldots \hat{\phi}(p_N) U^z_\mu(\Gamma(p_1, \ldots, p_N)) \, dp_1 \ldots dp_N \]

\[ U(\Gamma(p_1, \ldots, p_N)) = \int d^D k_1 \ldots d^D k_L \, I_\Gamma(k_1, \ldots k_L, p_1, \ldots p_N) \]

\[ U^z_\mu(\Gamma(p_1, \ldots, p_N)) : \text{DimReg+MS} \]

\[ = \int \mu^{zL} d^{D-z} k_1 \ldots d^{D-z} k_L \, I_\Gamma(k_1, \ldots k_L, p_1, \ldots p_N) \]

Laurent series in \( z \)
BPHZ renormalization scheme

Class of subgraphs $\mathcal{V}(\Gamma)$:

$\mathcal{T}$ renormalizable theory, $\Gamma = 1\text{PI}$ Feynman graph: $\mathcal{V}(\Gamma)$ (not necessarily connected) subgraphs $\gamma \subset \Gamma$ with

1. Edges of $\gamma$ are internal edges of $\Gamma$.

2. Let $\tilde{\gamma}$ be a graph obtained by adjoining to a connected component of $\gamma$ the edges of $\Gamma$ that meet the component. Then $\tilde{\gamma}$ is a Feynman graph of the theory $\mathcal{T}$.

3. The unrenormalized value $U(\tilde{\gamma})$ is divergent.

4. The graph $\Gamma/\gamma$ is a Feynman graph of the theory.

5. The components of $\gamma$ are 1PI graphs.

6. The graph $\Gamma/\gamma$ is a 1PI graph.
BPHZ procedure:

Preparation:

\[
\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)
\]

Coefficient of the pole part is given by a local term

Counterterms:

\[
C(\Gamma) = -T(\overline{R}(\Gamma))
\]

\[
= -T \left( U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma) \right)
\]

\(T = \) projection on the polar part of the Laurent series

Renormalized value:

\[
R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma)
\]

\[
= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)
\]
Connes-Kreimer Hopf algebra of Feynman graphs

**Discrete version** (over $k = \mathbb{C}$, in fact $k = \mathbb{Q}$)

$\mathcal{H} = \mathcal{H}(\mathcal{I})$ depends on the theory $\mathcal{I}$

Generators: 1PI graphs $\Gamma$ of the theory

Grading: $\deg(\Gamma_1 \cdots \Gamma_r) = \sum_i \deg(\Gamma_i)$
and $\deg(1) = 0$

Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in V(\Gamma)} \gamma \otimes \Gamma / \gamma$$

Antipode: inductively (lower deg)

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$
Affine group scheme $G(\mathcal{H}(T)) = \text{Difg}(T)$

“diffeographisms”

$$\text{Difg}(T) \rightarrow \text{Diff}$$

to formal diffeomorphisms of the coupling constants

$$g_{\text{eff}} = g + \sum_{n} \alpha_n g^n, \quad \alpha_n \in \mathcal{H}$$

Lie algebra: (Milnor-Moore)

$$[\Gamma, \Gamma'] = \sum_v \Gamma \circ_v \Gamma' - \sum_{v'} \Gamma' \circ_{v'} \Gamma$$

$\Gamma \circ_v \Gamma' =$ inserting $\Gamma'$ in $\Gamma$ at the vertex $v$

**Continuous version** On $E_\Gamma := \{(p_i)_{i=1,...,N} ; \sum p_i = 0\}$ distributions

$$C_c^{-\infty}(E) = \bigoplus_{\Gamma} C_c^{-\infty}(E_\Gamma)$$

Hopf algebra

$$\tilde{\mathcal{H}}(T) = \text{Sym}(C_c^{-\infty}(E))$$

$$\Delta(\Gamma, \sigma) = (\Gamma, \sigma) \otimes 1 + 1 \otimes (\Gamma, \sigma) + \sum_{\gamma \in \mathcal{V}(T); i \in \{0,1\}} (\gamma_i, \sigma_i) \otimes (\Gamma / \gamma_i, \sigma)$$
Loops and Birkhoff factorization

\[ \Delta = \text{(infinitesimal) disk around } z = 0, \ C = \partial \Delta \]
\[ C_+ \cup C_- = \mathbb{P}^1(\mathbb{C}) \setminus C \]
\[ G(\mathbb{C}) = \text{complex connected Lie group} \]
loop \( \gamma : C \to G(\mathbb{C}) \)

Birkhoff factorization problem: is it possible to factor

\[ \gamma(z) = \gamma_-(z)^{-1} \gamma_+(z) \]
\[ \forall z \in C, \text{ with } \gamma_\pm : C_\pm \to G(\mathbb{C}) \text{ holomorphic, } \gamma_-(\infty) = 1 \]

In general \textbf{no}: for \( G(\mathbb{C}) = \text{GL}_n(\mathbb{C}) \) only

\[ \gamma(z) = \gamma_-(z)^{-1} \lambda(z) \gamma_+(z) \]
\( \lambda(z) \) diagonal \( (z^{k_1}, z^{k_2}, \ldots, z^{k_n}) \): nontrivial holomorphic vector bundles on \( \mathbb{P}^1(\mathbb{C}) \) with \( c_1(L_i) = k_i \) and

\[ E = L_1 \oplus \ldots \oplus L_n \]
$\mathcal{H}$ commutative Hopf algebra over $\mathbb{C}$:

$K = \mathbb{C}\{z\} = \mathbb{C}\{z\}[z^{-1}]$, $\mathcal{O} = \mathbb{C}\{z\}$, $\mathcal{Q} = z^{-1}\mathbb{C}[z^{-1}]$, $\tilde{\mathcal{Q}} = \mathbb{C}[z^{-1}]$

loop $\gamma(z)$: element $\phi \in G(K) = \text{Hom}_{\mathcal{A}\mathbb{C}}(\mathcal{H}, K)$

positive part $\gamma_+(z)$: element $\phi_+ \in G(\mathcal{O})$

negative part $\gamma_-(z)$: element $\phi_- \in G(\tilde{\mathcal{Q}})$

$\gamma_-(\infty) = 1 \iff \varepsilon_- \circ \phi_- = \varepsilon$

Birkhoff $\gamma(z) = \gamma_-(z)^{-1} \gamma_+(z)$ becomes

$\phi = (\phi_- \circ S) \ast \phi_+$

Product $\phi_1 \ast \phi_2$ dual to coproduct

$\langle \phi_1 \ast \phi_2, X \rangle = \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$
$G = \text{pro-unipotent affine group scheme of a commutative Hopf algebra } \mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$

Always have Birkhoff factorization: inductive formula (CK)

$$\phi_-(X) = -T \left( \phi(X) + \sum \phi_-(X') \phi(X'') \right)$$

$$\phi_+(X) = \phi(X) + \phi_-(X) + \sum \phi_-(X') \phi(X'')$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

\textbf{BPHZ = Birkhoff} Take $G = \tilde{\text{Difg}}(T)$ (continuous version)

Data $U^z(\Gamma(p_1, \ldots, p_N))$: homomorphism $U : \tilde{\mathcal{H}}(T) \to K$

$$(\Gamma, \sigma) \mapsto h(z) = \langle \sigma, U^z(\Gamma(p_1, \ldots, p_N)) \rangle$$

Laurent series

$\phi = U, \phi_- = C, \phi_+ = R$: same as BPHZ!
Dependence on mass scale: $\gamma_\mu(z)$

$$\gamma_\mu(z) = \gamma_{\mu-}(z)^{-1}\gamma_{\mu+}(z)$$

Grading by loop number:

$Y(X) = nX$, $\forall X \in H_n^\vee(T)$

$$\theta_t \in \text{Aut}(\text{Difg}(T)), \quad \frac{d}{dt} \theta_t|_{t=0} = Y$$

Main properties of scale dependence:

$$(*) = \begin{cases} 
\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)) \\
\frac{\partial}{\partial \mu} \gamma_{\mu-}(z) = 0.
\end{cases}$$

Renormalization group:

$$F_t = \lim_{z \to 0} \gamma_-(z) \theta_{tz}(\gamma_-(z)^{-1})$$

action $\gamma_{e^t\mu}(0) = F_t \gamma_{\mu+}(0)$

Beta function: $\beta = \frac{d}{dt} F_t|_{t=0} \in \mathfrak{g}$

$$\beta := Y \text{ Res } \gamma, \quad \text{Res}_{\gamma=0} \gamma := - \left( \frac{\partial}{\partial u} \gamma_-(\frac{1}{u}) \right)_{u=0}$$
Connes-Kreimer theory in a nutshell:

\[ G = \text{pro-unipotent affine group scheme} \ (\cong \text{Difg}(T)) \]

\[ L(G(\mathbb{C}), \mu) = \text{loops } \gamma_\mu(z) \text{ with (*) properties} \]

Divergences (counterterms) \[ \gamma_-(z) \]

Renormalized values \[ \gamma_\mu^+(0) \]

⇒ Understand data \[ L(G(\mathbb{C}), \mu) \text{ and } \gamma_-(z) \]
Renormalization and the Riemann-Hilbert correspondence (AC–MM)

Tannakian formalism

Abelian category $\mathcal{C}$:

- $\text{Hom}_\mathcal{C}(X, Y)$ abelian groups
  ($\exists 0 \in \text{Obj}(\mathcal{C})$ with $\text{Hom}_\mathcal{C}(0, 0)$ trivial group)

- There are products and coproducts: $\forall X, X' \in \text{Obj}(\mathcal{C})$, $\exists Y \in \text{Obj}(\mathcal{C})$ and
  \[ X \xrightarrow{f_1} Y \xleftarrow{f_2} X' \quad \text{and} \quad X \xleftarrow{h_1} Y \xrightarrow{h_2} X', \]
  with $h_1f_1 = 1_X$, $h_2f_2 = 1_{X'}$, $h_2f_1 = 0 = h_1f_2$, $f_1h_2 + f_2h_1 = 1_Y$.

- There are Kernels and Cokernels: $\forall X, Y \in \text{Obj}(\mathcal{C})$, $\forall f : X \to Y$ can decompose $j \circ i = f$,
  \[ K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K', \]
  with $K = \text{Ker}(f)$, $K' = \text{Coker}(f)$, and $I = \text{Ker}(k) = \text{Coker}(c)$.
$k$-linear category $C$: $\text{Hom}_C(X, Y)$ is a $k$-vector space $\forall X, Y \in \text{Obj}(C)$.

Tensor category $C$: $k$-linear with $\otimes : C \times C \to C$

- $\exists 1 \in \text{Obj}(C)$ with $\text{End}(1) \cong k$ and functorial isomorphisms
  
  $$a_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$
  
  $$c_{X,Y} : X \otimes Y \to Y \otimes X$$
  
  $$l_X : X \otimes 1 \to X \quad \text{and} \quad r_X : 1 \otimes X \to X.$$  

- Commutativity: $c_{Y,X} = c^{-1}_{X,Y}$

Rigid tensor category $C$: tensor with duality $\vee : C \to C^{op}$

- $\forall X \in \text{Obj}(C)$ the functor $- \otimes X^\vee$ is left adjoint to $- \otimes X$ and the functor $X^\vee \otimes -$ is right adjoint to $X \otimes -$.

- Evaluation morphism $\epsilon : X \otimes X^\vee \to 1$ and unit morphism $\delta : 1 \to X^\vee \otimes X$ with $(\epsilon \otimes 1) \circ (1 \otimes \delta) = 1_X$ and $(1 \otimes \epsilon) \circ (\delta \otimes 1) = 1_{X^\vee}$.  

**Functors** $\omega : \mathcal{C} \to \mathcal{C}'$

faithful: $\omega : \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_{\mathcal{C}'}(\omega(X), \omega(Y))$ injection

additive: $\omega : \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_{\mathcal{C}'}(\omega(X), \omega(Y))$ $k$-linear

exact: $0 \to X \to Y \to Z \to 0$ exact $\Rightarrow 0 \to \omega(X) \to \omega(Y) \to \omega(Z) \to 0$ exact

tensor: functorial isomorphisms $\tau_1 : \omega(1) \to 1$ and $\tau_{X,Y} : \omega(X \otimes Y) \to \omega(X) \otimes \omega(Y)$

**Fiber functor, Tannakian categories** $\mathcal{C}$ be a $k$-linear rigid tensor category: fiber functor $\omega : \mathcal{C} \to \text{Vect}_K$ exact faithful tensor functor, $K$ extension of $k$.

$\Rightarrow \mathcal{C}$ Tannakian (=has fiber functor), neutral Tannakian ($K = k$)

(Grothendieck, Savendra-Rivano, Deligne, . . . )

$\mathcal{C}$ neutral Tannakian $\Rightarrow \mathcal{C} \cong \text{Rep}_G$

$G = \text{Aut}^\otimes(\omega)$ affine group scheme $\text{Gal}(\mathcal{C})$
Example: \( \text{Rep}_Z \cong \text{Rep}_G \) affine group scheme \( G = \hat{\mathbb{Z}} \) dual to \( \mathcal{H} = \mathbb{C}[e(q), t] \), for \( q \in \mathbb{C}/\mathbb{Z} \), with relations \( e(q_1 + q_2) = e(q_1)e(q_2) \) and coproduct \( \Delta(e(q)) = e(q) \otimes e(q) \) and \( \Delta(t) = t \otimes 1 + 1 \otimes t \).

**Riemann–Hilbert correspondence**

Tannakian formalism applied to categories of differential systems (differential Galois theory)

\( (K, \delta) = \) differential field
e.g. \( K = \mathbb{C}\{z\}[z^{-1}] \) or \( K = \mathbb{C}((z)) \)

Category \( \mathcal{D}_K \) of differential modules over \( K \):
- Objects \( (V, \nabla) \), vector space \( V \in Obj(\mathcal{V}_K) \) and connection
  - \( \mathbb{C} \)-linear map \( \nabla : V \to V \) with \( \nabla(fv) = \delta(f)v + f\nabla(v) \), for all \( f \in K \) and all \( v \in V \)
- Morphisms \( \text{Hom}((V_1, \nabla_1), (V_2, \nabla_2)) \) \( K \)-linear maps
  - \( T : V_1 \to V_2 \) with \( \nabla_2 \circ T = T \circ \nabla_1 \)

\[ (V_1, \nabla_1) \otimes (V_2, \nabla_2) = (V_1 \otimes V_2, \nabla_1 \otimes 1 + 1 \otimes \nabla_2) \]
and dual \( (V, \nabla)^\vee \)
Fiber functor $\omega(V, \nabla) = \text{Ker}\nabla$. Neutral Tannakian category $\mathcal{D}_K \cong \text{Rep}_G$

For $K = \mathbb{C}((z))$, affine group scheme $G = T \times \tilde{\mathbb{Z}}$ of Ramis exponential torus $T = \text{Hom}(B, \mathbb{C}^*)$ with $B = \bigcup_{\nu \in \mathbb{N}} B_{\nu}$, for $B_{\nu} = z^{-1/\nu} \mathbb{C}[z^{-1/\nu}]$.

For $K = \mathbb{C}\{z\}[z^{-1}]$ extra generators: Stokes phenomena (resummation of divergent series in sectors)

Example: ODE $\delta(u) = Au$, subcategory of $\mathcal{D}_K \Rightarrow$ differential Galois group (Aut of Picard-Vessiot ring)

Example: ODE $\delta(u) = Au$ regular-singular iff $\exists T$ invertible matrix coeff. in $K = \mathbb{C}((z))$, with $T^{-1}AT - T^{-1}\delta(T) = B/z$, $B$ coeff. in $\mathbb{C}[[z]]$. Tannakian subcategory $\mathcal{D}^{rs}_K$ of $\mathcal{D}_K$ gen. by regular-singular equations $\mathcal{D}^{rs}_K \cong \text{Rep}_{\tilde{\mathbb{Z}}}$ (monodromy $\mathbb{Z} = \pi_1(\Delta^*)$)
**Claim:** There is a Riemann-Hilbert correspondence associated to the data of perturbative renormalization

- Not just over the disk \( \Delta \) but a \( \mathbb{C}^* \)-fibration \( B \) over \( \Delta \), so we exit from the category \( \mathcal{D}_K \).

- Equivalence relation on connections by gauge transformations regular at \( z = 0 \).

- Class of connections (equisingular connections) not regular-singular: setting of “irregular” Riemann–Hilbert correspondence with arbitrary degree of irregularity, as for \( \mathcal{D}_K \).

- The Galois group same in formal and non-formal case (no Stokes phenomena).
Data of CK revisited

\( G = \) pro-unipotent affine group scheme (\( = \text{Difg}(\mathcal{T}) \))

\( L(G(\mathbb{C}), \mu) = \) loops \( \gamma_\mu(z) \) with

\[
(*) = \begin{cases} 
\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)) \\
\frac{\partial}{\partial \mu} \gamma_\mu^-(z) = 0.
\end{cases}
\]

Divergences (counterterms) \( \gamma^-(z) \)

First step (CK):

\[
\gamma^-(z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n}
\]

coefficients \( d_n \in \mathcal{H}^\vee \)

\[
Y d_{n+1} = d_n \beta \quad \forall n \geq 1, \quad \text{and} \quad Y d_1 = \beta
\]

\( \Rightarrow \) Can write as iterated integrals
Time ordered exponential

g(\mathbb{C})\text{-valued smooth } \alpha(t), \ t \in [a, b] \subset \mathbb{R}

\mathcal{T}e\int_{a}^{b} \alpha(t) \, dt := 1 + \sum_{1}^{\infty} \int_{a}^{b} \alpha(s_1) \cdots \alpha(s_n) \, ds_1 \cdots ds_n

product in \mathcal{H}^\vee, \ with \ 1 \in \mathcal{H}^\vee \ \text{counit } \varepsilon \ \text{of } \mathcal{H}

- Paired with \ X \in \mathcal{H} \ \text{the sum is finite.}

- Defines an element of \ G(\mathbb{C}).

- Value \ g(b) \ of unique solution \ g(t) \in G(\mathbb{C}) \ with \ g(a) = 1 \ of

\[ dg(t) = g(t) \alpha(t) \, dt \]

- Multiplicative over sum of paths:

\[ \mathcal{T}e\int_{c}^{a} \alpha(t) \, dt = \mathcal{T}e\int_{a}^{b} \alpha(t) \, dt \mathcal{T}e\int_{b}^{c} \alpha(t) \, dt \]
\[ \gamma_\mu(z) \in L(G(\mathbb{C}), \mu), \text{ then} \]
\[ \gamma_-(z) = T e^{-\frac{1}{z}} \int_0^\infty \theta_{-t}(\beta) dt \]

by \( \gamma_-(z)^{-1} = 1 + \sum_{n=1}^\infty \frac{d_n}{z^n} \) with
\[ d_n = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta) \theta_{-s_2}(\beta) \cdots \theta_{-s_n}(\beta) \, ds_1 \cdots ds_n \]

\[ \gamma_\mu(z) \in L(G(\mathbb{C}), \mu), \text{ then} \]
\[ \gamma_\mu(z) = T e^{-\frac{1}{z}} \int_0^{-z \log \mu} \theta_{-t}(\beta) dt \, \theta_z \log \mu(\gamma_{\text{reg}}(z)) \]
for a unique \( \beta \in g(\mathbb{C}) \) (with \( \gamma_{\text{reg}}(z) \) a loop regular at \( z = 0 \))

The Birkhoff factorization
\[ \gamma_\mu^+(z) = T e^{-\frac{1}{z}} \int_0^{-z \log \mu} \theta_{-t}(\beta) dt \, \theta_z \log \mu(\gamma_{\text{reg}}(z)) \]
\[ \gamma_-(z) = T e^{-\frac{1}{z}} \int_0^\infty \theta_{-t}(\beta) dt \]

Conversely, given \( \beta \in g(\mathbb{C}) \) and \( \gamma_{\text{reg}}(z) \) regular
\[ \Rightarrow \gamma_\mu \in L(G(\mathbb{C}), \mu) \]
\[ \varpi = \alpha(s, t) ds + \eta(s, t) dt \] is flat \( \mathfrak{g}(\mathbb{C}) \)-valued connection

\[ \partial_s \eta - \partial_t \alpha + [\alpha, \eta] = 0 \]

\[ \mathbb{T} e^{\int_0^1 \gamma^* \varpi} \] depends on homotopy class of path

Differential field \((K, \delta)\) with \( \text{Ker} \delta = \mathbb{C} \)

log derivative on \( G(K) \)

\[ D(f) := f^{-1} f' \in \mathfrak{g}(K) \]

\[ f'(X) = \delta(f(X)), \quad \forall X \in H \]

Differential equation \( D(f) = \varpi \)

Existence of solutions: trivial monodromy

\[ G = \lim_{\leftarrow i} G_i, \text{ monodromy} \]

\[ M_i(\varpi)(\gamma) := \mathbb{T} e^{\int_0^1 \gamma^* \varpi} \]

punctured disk \( \Delta_i^* \) of positive radius

\[ M(\varpi) = 1 \]

well defined on \( G \)
\((K, \delta), \ d : K \rightarrow \Omega^1, \ df = \delta(f) \, dz\)

\[D : G(K) \rightarrow \Omega^1(\mathcal{g}), \quad Df = f^{-1} \, df\]

\[D(fh) = Dh + h^{-1} \, Df \, h\]

Two connections \(\varpi\) and \(\varpi'\) are equivalent iff

\[\varpi' = Dh + h^{-1} \varpi \, h, \quad \text{with} \quad h \in G(\mathcal{O})\]

Equivalent \(\Leftrightarrow\) same negative part of Birkhoff:

\(D(f\varpi) = \varpi\) and \(D(f\varpi') = \varpi'\) solutions in \(G(K)\)

\[\varpi \sim \varpi' \iff f\varpi = f\varpi'\]
**Flat equisingular connections:** accounts for $\mu$-dependence

Principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle $\mathbb{G}_m \to B \xrightarrow{\pi} \Delta$ over infinitesimal disk $\Delta$.

$P = B \times G$, $P^* = P|_{B^*}$, $B^* = B|_{\Delta}$.

Action of $\mathbb{G}_m$ by $b \mapsto u(b)$, $\forall u \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ and action of $\mathbb{G}_m$ on $G$ dual to graded Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$

$$u(b, g) = (u(b), u^Y(g)), \quad \forall u \in \mathbb{G}_m$$

Flat connection $\varpi$ on $P^*$ is **equisingular** iff

- $\varpi$ is $\mathbb{G}_m$-invariant
  $$\varpi(z, u(v)) = u^Y(\varpi(z, v)), \quad \forall u \in \mathbb{G}_m$$

- $v = (\sigma(z), g)$, for $z \in \Delta$ and $g \in G$

- all the restrictions are equivalent
  $$\sigma_1^*(\varpi) \sim \sigma_2^*(\varpi)$$

$\sigma_1$ and $\sigma_2$ are two sections of $B$ as above, with $\sigma_1(0) = y_0 = \sigma_2(0)$

The connections $\sigma_1^*(\varpi)$ and $\sigma_2^*(\varpi)$ have the same type of singularity at the origin $z = 0$
Equivalence: $\varpi$ and $\varpi'$ on $P^*$ equivalent iff

$$\varpi' = Dh + h^{-1}\varpi h,$$

with $h$ a $G$-valued $\mathbb{G}_m$-invariant map regular in $B$.

**Thm:** Bijective correspondence between equivalence classes of flat equisingular $G$-connections $\varpi$ on $P^*$ and elements $\beta \in \mathfrak{g}(\mathbb{C})$

$\varpi \sim D\gamma$ with

$$\gamma(z, v) = Te^{-\frac{1}{z} \int_0^v u^Y(\beta) \frac{du}{u}}$$

(integral on the path $u = tv$, $t \in [0, 1]$)

Correspondence independent of choice of section $\sigma : \Delta \to B$ with $\sigma(0) = y_0$.

**Key step:** vanishing of monodromies around $\Delta^*$ and $\mathbb{C}^*$
Category of equivariant flat vector bundles

\[ V = \bigoplus_{n \in \mathbb{Z}} V_n \] fin dim \( \mathbb{Z} \)-graded vector space; trivial vector bundle \( E = B \times V \) filtered by \[ W^{-n}(V) = \bigoplus_{m \geq n} V_m \]
\( \mathbb{G}_m \) action induced by grading.

\( W \)-connection on a filtered vector bundle \((E, W)\) over \( B \):

\[ W^{-n-1}(E) \subset W^{-n}(E), \]
\[ Gr_n^W(E) = W^{-n}(E)/W^{-n-1}(E) \]

Connection \( \nabla \) on \( E^* = E|_{B^*} \), compatible with filtration: restricts to \( W^{-n}(E^*) \) and induces trivial connection on \( Gr^W(E) \)

Two \( W \)-connections \( \nabla_i \) on \( E^* \) are \( W \)-equivalent iff \( \exists T \in \text{Aut}(E) \), preserving filtration, inducing identity on \( Gr^W(E) \), with \( T \circ \nabla_1 = \nabla_2 \circ T \)

A \( W \)-connection \( \nabla \) on \( E \) is equisingular if it is \( \mathbb{G}_m \)-invariant and all restrictions to sections \( \sigma : \Delta \to B \) with \( \sigma(0) = y_0 \) are \( W \)-equivalent.
Category $\mathcal{E}$ equisingular flat vector bundles

$\text{Obj}(\mathcal{E})$ pairs $\Theta = (V, [\nabla])$

$V = \text{fin dim } \mathbb{Z}\text{-graded vector space}, [\nabla] = W\text{-equivalence class of flat equisingular } W\text{-connection } \nabla$ on $E^* = B^* \times V$

Morphisms: $T \in \text{Hom}_\mathcal{E}(\Theta, \Theta')$ linear map $T : V \to V'$
compatible with the grading and on $(E' \oplus E)^*$

$$\nabla_1 = \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix}$$

$$\nabla_2 = \begin{pmatrix} \nabla' & T \nabla - \nabla' T \\ 0 & \nabla \end{pmatrix}$$

are $W$-equivalent on $B$

(Notice: category of filtered vector spaces, with morphisms linear maps respecting filtration, is not an abelian category)
For $G = \text{Difg}(T)$, $\varpi = \text{flat equisingular connection on}$ $P^* = B^* \times G$, fin dim lin rep $\xi : G \rightarrow \text{GL}(V) \Rightarrow \Theta \in \text{Obj}(\mathcal{E})$. Equivalent $\varpi$ give same $\Theta$.

**THM** The category $\mathcal{E}$ is a neutral Tannakian category (over $\mathbb{C}$, over $\mathbb{Q}$) with fiber functor $\omega(\Theta) = V$

$$\mathcal{E} \cong \text{Rep}_{U^*}$$

$U^* = U \times \mathbb{G}_m$ affine group scheme, $U = \text{prounipotent dual to Hopf algebra}$

$$\mathcal{H}_U = U(\mathcal{L}_U)^\vee$$

$\mathcal{L}_U = \mathcal{F}(1, 2, 3, \cdots)$ denote the free graded Lie algebra generated by elements $e_{-n}$ of degree $n$, for each $n > 0$
Renormalization group

\[ e = \sum_{1}^{\infty} e^{-n} \]

determines \( \text{rg} : \mathcal{G}_a \to \mathbb{U} \)

Universal singular frame

\[ \gamma_U(z, v) = \mathcal{T} e^{-\frac{1}{z} \int_{0}^{v} u^{Y(e)} \frac{du}{u}} \]

Universal source of counterterms

Coefficients:

\[ \gamma_U(z, v) = \sum_{n \geq 0} \sum_{k_j > 0} \frac{e_{-k_1} e_{-k_2} \cdots e_{-k_n}}{k_1 (k_1 + k_2) \cdots (k_1 + k_2 + \cdots + k_n)} v^j z^{-n} \]

(local index formula Connes-Moscovici)
Key step in proof of THM: for $\Theta = [V, \nabla]$ be an object of $\mathcal{E}$, there exists a unique representation $\rho = \rho_\Theta$ of $\mathbb{U}^*$ in $V$, such that

$$D\rho(\gamma_U) \simeq \nabla$$

universal singular frame $\gamma_U$

Note: $Q(n) \in \text{Obj}(\mathcal{E})$ with $V$ 1-dim over $\mathbb{Q}$ in deg $n$, $\nabla$ trivial connection on assoc bundle $E$ over $B$. Fiber functor:

$$\omega_n(\Theta) = \text{Hom}(Q(n), \text{Gr}^W_{-n}(\Theta))$$
For $G = \text{Difg}(\mathcal{T})$, canonical bijection: equivalence classes of flat equisingular connections on $P^*$ and graded representations

$$\rho : \mathbb{U}^* \to G^* = G \rtimes \mathbb{G}_m$$

Using the beta function:

$$\beta = \sum_{1}^{\infty} \beta_n$$

$Y(\beta_n) = n\beta_n$, representation $\mathbb{U} \to G$ compatible with $\mathbb{G}_m$:

$$e^{-n} \mapsto \beta_n$$

Action on physical constants through $\text{Difg} \to \text{Diff}$ map:

$$\mathbb{U} \to \text{Difg}(\mathcal{T}) \to \text{Diff}$$
Motives

Cohomologies for alg varieties:

de Rham $H^{\cdot}_{dR}(X) = \mathbb{H}^{\cdot}(X, \Omega^{\cdot}_X)$
Betti $H^{\cdot}_B(X, \mathbb{Q})$ (singular homology)
étale $H^{\iota}_{et}(\bar{X}, \mathbb{Q}_\ell)$ for $\ell \neq \text{char } k$ and $\bar{X}$ over $\bar{k}$.

Isomorphisms: period isomorphism

$$H^{i}_{dR}(X, k) \otimes \sigma \mathbb{C} \cong H^{i}_B(X, \mathbb{Q}) \otimes \mathbb{Q} \mathbb{C}$$

and comparison isom

$$H^{i}_B(X, \mathbb{Q}) \otimes \mathbb{Q} \mathbb{Q}_\ell \cong H^{i}_{et}(\bar{X}, \mathbb{Q}_\ell)$$

Universal cohomology theory? Motives

Linearization of the category of algebraic varieties (adding morphisms; analog with Morita theory for algebras)

$$X \mapsto h(X) = \bigoplus_i h^i(X)$$

if $h^j = 0, \forall j \neq i$, pure of weight $i$

Pure motives “direct summands of algebraic varieties”
Pure Motives

Objects \((X, p)\), \(p = p^2 \in \text{End}(X)\), \(X\) smooth projective

Morphisms \(\text{Hom}(X, Y)\) correspondences: alg cycles in \(X \times Y\), \(\text{codim} = \dim X\). Equivalences (numerical, rational,...)
\[\text{Hom}((X, p), (Y, q)) = q\text{Hom}(X, Y)p\]

Tate motives \(\mathbb{Q}(1)\) inverse of \(h^2(\mathbb{P}^1)\), \(\mathbb{Q}(0) = h(pt)\), \(\mathbb{Q}(n + m) = \mathbb{Q}(n) \otimes \mathbb{Q}(m)\)

(Grothendieck standard conjectures)
Jannsen: numerical equivalence \(\Rightarrow\) neutral Tannakian category (fiber functor Betti cohomology) \(\Rightarrow\)
\(\text{Rep}_G\) affine group scheme \(G\)

Tate motives \(G = \mathbb{G}_m\).
Mixed motives

Extend “universal cohomology theory” to $X$ not smooth projective: technically much more complicated, via constructions of derived category (Voevodsky, Levine, Hanamura)

Mixed Tate motives
(filtered: graded pieces Tate motives)

Full subcategory of Tate motives (over a field $k$ or a scheme $S$) $\mathcal{MT}_{mix}(S)$ (Deligne–Goncharov)

Motivic Galois group of $\mathcal{MT}_{mix}(k)$ extension $G \times \mathbb{G}_m$, $G$ pro-unipotent, $\text{Lie}(G)$ free one generator in each odd degree $n \leq -3$

$\textbf{THM}$(CM) (non-canonical) isomorphism $U^* \sim G_{\mathcal{MT}}(\mathcal{O})$ with motivic Galois group of the scheme $S_4$ of 4-cyclotomic integers

$\mathcal{O} = \mathbb{Z}[i][1/2]$