**Def.** Contragradient bimodule $M^0$

of a bimodule $M$ over $A_{LR}$

$$M^0 = \{ \tilde{\gamma} : \tilde{\gamma} \in M \} \text{ with action}$$

$$a \tilde{\gamma} b := b^* \tilde{\gamma} a^* \quad a, b \in A_{LR}$$

Then consider all possible inequivalent irreducible odd $A_{LR}$ - bimodules

$$M_F = \text{direct sum of all these}$$

**Prop:** $\dim_M M_F = 32$

$$M_F = \mathcal{E} \oplus \mathcal{E}^0$$

$$\mathcal{E} = 2_L \otimes 1^0 \oplus 1^0 \otimes 2_R \oplus 1^0 \otimes 2_L \otimes 3^0 \oplus 2_R \otimes 3^0$$

Isomorphism (anti-linear)

$$J_F : M_F \to M_F$$

$$J_F (\tilde{\gamma}, \tilde{\eta}) = (\tilde{\eta}, \tilde{\gamma}) \quad \forall \tilde{\gamma}, \tilde{\eta} \in M_F$$

Satisfies $J_F^2 = 1$ and $\tilde{\gamma} b = J_F b^* J_F \tilde{\gamma} \quad \forall \tilde{\gamma}, b \in M_F$
Sign: $\gamma_F \in \mathbb{Z}/2\mathbb{Z}$ - grading on $M_F$

given by

$$\gamma_F = c - J_F \circ J_F$$

where $c = (0, +1, -1, 0) \in \mathbb{A}_L$ (chirality)

Notice: $\gamma_F$ and $J_F$ satisfy relations

$$J_F^2 = 1 \quad J_F \gamma_F = -\gamma_F J_F$$

$$\Rightarrow \quad \epsilon = 1 \quad \epsilon'' = -1 \quad \Rightarrow \quad n = 6 \mod 8$$

This finite dim. alg. (i.e. physically zero dimensional space) is 6-dimensional from the point of view of KO-dimension!

Generations $\quad \text{Input} \quad N$

choose $N = 3$ have models for other choices of $N$
this is not deduced from previous input
but assigned as additional input

Comment: The model does not predict the number of generations but there are reasons (see later) why $N = 3$ is an especially nice choice in this type of models
\[ N = 3 \text{ generations} \]

\[ H_f = \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3} \quad \bar{H}_f = \mathbf{\bar{3}} \oplus \mathbf{\bar{3}} \oplus \mathbf{\bar{3}} \]

\[ H_F = H_f \oplus \bar{H}_f = M_F \oplus M_F \oplus M_F \]

Consider the left action of \( A_{LR} \) on \( H_F \)

\[ \rho(a) = \pi(a) \oplus \pi'(a) \]

acting on \( H_f \) and \( \bar{H}_f \)

(\text{action of the algebra does not mix matter and antimatter})

\( \pi, \pi' \) "disjoint" i.e. no equivalent subrepresentations

Can see directly from

\[ E = L_L \otimes 1^0 + L_R \otimes 1^0 + L_L \otimes 3^0 + L_R \otimes 3^0 \]

More explicit description of \( H_F \) and \( A_{LR} \)

Basis for \( H_F \) and physical meaning in terms of particles.
2 repres of #1 by \( q = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \)

basis \( \uparrow \) and \( \downarrow \):

\( \lambda \in \text{C} \subset \mathcal{H} \)

\( q(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \) acts on \( \uparrow \) by

\( q(\lambda) \uparrow = \lambda \uparrow \)

and on \( \downarrow \) by

\( q(\lambda) \downarrow = \bar{\lambda} \downarrow \)

Use notation

\[ u, \bar{u}, u_1, \bar{u}_1, u_2, \bar{u}_2, u_3, \bar{u}_3 \]

\[ u \in \uparrow \otimes \mathbb{B}^o \quad \text{C } \mathbb{R} \otimes \mathbb{B}^o \]

\[ \bar{u} \in \mathbb{B}^o \otimes \uparrow \quad \text{C } \mathbb{B}^o \otimes \mathbb{R}^o \]

\( u = u_i \quad i = 1, 2, 3 \)

index in \( \mathbb{B}^o \) (color index)

\( \bar{u} = \bar{u}_j \quad j = 1, 2, 3 \)

Since have two copies \( L \) and \( R \)

use notation

\[ \uparrow L, \uparrow R \quad \text{and } u_L, u_R, \bar{u}_L, \bar{u}_R \]

\[ d_L, d_R, \bar{d}_L, \bar{d}_R \]

where similarly define

\[ d, \bar{d}, \text{ fn } \quad d \in \downarrow \otimes \mathbb{B}^o \quad \text{C } 2 \otimes \mathbb{B}^o \]

\[ \bar{d} \in \mathbb{B}^o \otimes \downarrow \quad \text{C } \mathbb{B}^o \otimes \mathbb{R}^o \]
Also use notation

\[ \emptyset \text{ for } |\uparrow\rangle \otimes 1^\circ \subset 2 \otimes 1^\circ \]
\[ \overline{1} \text{ for } \begin{array}{c}
\ldots \\
\end{array} |\downarrow\rangle \otimes 1^\circ \subset 1 \otimes 2^\circ \\
\text{and}
\]
\[ e \text{ for } |\uparrow\rangle \otimes 1^\circ \subset 2 \otimes 1^\circ \]
\[ \bar{e} \text{ for } \begin{array}{c}
\ldots \\
\end{array} |\downarrow\rangle \otimes 1^\circ \subset 1 \otimes 2^\circ \]

\[ \begin{array}{c}
\nu_L \bar{\nu}_L \nu_R \bar{\nu}_R \\
e_L \bar{e}_L e_R \bar{e}_R \\
U_{L,R} \bar{U}_{L,R} \text{ up quarks (with extra generation index } \lambda = 1, \ldots, N \text{)}
\\d_{L,R} \bar{d}_{L,R} \text{ down quarks}
\\e_{L,R} \bar{e}_{L,R} \text{ charged leptons (electron, muon, tau)}
\\\nu_{L,R} \bar{\nu}_{L,R} \text{ neutrinos (with right handed neutrinos!)}
\end{array} \]

Then action of \( A_{LR} \): \[ a = (\lambda, q, q_R, m) \]
on \[ \begin{array}{c}
H_f \end{array} \]
\[ a \left( u_L \right) = \left( \begin{array}{c}
\alpha u_L - \beta d_L \\
\beta u_L + \alpha d_L \\
\end{array} \right) \]
\[ a \left( \overline{u}_R \right) = \left( \begin{array}{c}
\alpha \overline{u}_R - \beta \overline{d}_R \\
\beta \overline{u}_R + \alpha \overline{d}_R \\
\end{array} \right) \]
\[ a \left( \nu_L \right) = \left( \begin{array}{c}
\nu_L \\
\nu_R \\
\end{array} \right) \]
\[ a \left( \overline{\nu}_R \right) = \left( \begin{array}{c}
\bar{\nu}_R \\
\overline{\nu}_L \\
\end{array} \right) \]
\[
\mathcal{H}^-_f \quad a = (1, q_L, q_R, \mu)
\]
\[
a \bar{f} = \lambda \bar{f} \quad \text{for } f \text{ lepton} \quad (\text{i.e. span of } \bar{e}_L, \bar{\nu}_L, \bar{e}_R, \bar{\nu}_R)
\]
\[
a \bar{f} = m \bar{f} \quad \text{for } f \text{ quark} \quad (\text{i.e. span of } \bar{u}_L, \bar{u}_R, \bar{d}_L, \bar{d}_R)
\]

\[
\gamma_F \bar{f}_L = \bar{f}_L \quad \gamma_F \bar{f}_R = -\bar{f}_R \\
\gamma_F \bar{f}_L = -\bar{f}_L \quad \gamma_F \bar{f}_R = \bar{f}_R
\]

Breaking of the left-right symmetry of the algebra.

Introducing the Dirac operator \(D\)

Finite dimensional \((A, \mathcal{H})\) so

\[
D^* = D \quad \text{and } [[D, a], b^0] = 0
\]

conditions

together with

\[
DJ_F = J_F D \quad (n = 6 \mod 8)
\]

Notice that action of \(A_{LR}\) and \(\gamma_F\) and \(J_F\)

\(\underline{\text{don't mix } H_f \text{ and } H_{\bar{f}}}\)

Need to look for \(D\) that mixes \(H_f\) and \(H_{\bar{f}}\)

(otherwise have completely separate matter/antimatter worlds without interaction)
$A_{LR}$ with a $D$ mixing $H_f$ and $H_f^-$ does not satisfy $[[D,A],b^*] = 0$ (no order one condition)

Look for solutions $(A,D)$ with

$A_{LR}$ subalgebra same $H_f, J_f, V_f$ and $D$ with $[[D,A],b^*] = 0, \forall a,b \in A$
mixing $H_f$ and $H_f^-$

Step 1: $A(T) := \{ b \in A_{LR} : \pi'(b) T = T \pi(b) \} \forall \pi'$

for a given linear map

$T : H_f \rightarrow H_f^-$

Lemma: $A_{LR}$ involutive subalgebra, unital

1) If $\pi|_A$ and $\pi'|_A$ disjoint (no equiv. subrepresentations)

$\Rightarrow \exists$ off-diagonal Dirac

only: $\begin{pmatrix} D_f & 0 \\ 0 & D_f^- \end{pmatrix}$

2) If $\exists D = \begin{pmatrix} D_f & D_{off} \\ D_{off}^* & D_f^- \end{pmatrix}$ with $D_{off}$ \neq 0 in $A_{LR}$

$\Rightarrow \exists$ pair $e, e'$ min projections in commutants of $\pi(A_{LR})$ and $\pi'(A_{LR})$ and $T$ s.t. $e^* T e = T \neq 0$ and $A C A_c A_T$. 
Proof:

1) First notice that \([D,a]\) commutes with all \(a \in A_{LR}\) (all \(a \in A\)) by order one condition

\[ [D,a] \in A' \text{ is commutant of } A \]

\[ \Rightarrow [D,a] \text{ also in } A' \text{ (conjugating by } T) \]

\[ \Rightarrow [D,a] \text{ cannot have an off diagonal term} \]

mixing \(H_f\) and \(H_f^-\)

since action of \(A\) diagonal and without equivalent subrepresentations and \((D,a) \in A'\)

\[ \Rightarrow \text{if } D \text{ has off-diag. term } D_{HF} \text{ then} \]

\[ [D_{HF}, a] = 0 \forall a \in A \text{ but this again means } D_{HF} = 0 \]

2) If \(\pi, \pi'\) not disjoint \(\exists T \in \mathbb{C} \) s.t. \(\pi \circ A \subseteq A'(\pi')\)

if \(x, x' \in \pi(A_{LR})' \text{ and } \pi'(A_{LR})' \text{ (commutants)} \)

then \(A'(\pi') \subseteq A'(xTx')\)

in fact \(\pi'(b)T = T\pi'(b) = \pi'(b)x'Tx = x'Tx \pi'(b)\)

for \(x, x'\) commuting resp. with \(\pi'(b)\) and \(\pi(b)\)

\(\exists\) partition of unity by projections

\(\exists\ e, e'\) projections in \(\pi(A_{LR})'\) and \(\pi'(A_{LR})'\)

s.t. \(e'Te = 0\)

so can assume \(T\) is of this form