Spectral action

$$\mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right) \sim \sum_{\beta \in \mathrm{DimSp}} f_\beta \Lambda^\beta ⨍ |D|^{-\beta} + f(0)\,\zeta_D(0) + o(1)$$

$$f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, dv$$

Gilkey's theorem

M smooth Riemannian mfld dim $= m$

metric g

$P =$ second order differential operator of the form

$$P = -\left(g^{\mu\nu}\, \partial_\mu \partial_\nu + A^\mu \partial_\mu + B\right)$$

acting on sections of a bundle $\mathcal{E}$

so $A, B \in \mathrm{End}(\mathcal{E})$ and $g^{\mu\nu}\partial_\mu\partial_\mu \cdot \mathrm{id}_E$

then $$\mathrm{Tr}\left(e^{-tP}\right) \sim \sum_{n \geq 0} t^{\frac{n-m}{2}} \int_M a_n(x, P)\, dv(x)$$

$$dv(x) = \sqrt{\det g_{\mu\nu}(x)}\; d^m x$$

$$P = \nabla^* \nabla - E \qquad E \in \mathrm{End}(\mathcal{E})$$

$$\nabla_\mu = \partial_\mu + \omega'_\mu \qquad \omega'_\mu = \tfrac{1}{2} g_{\mu\nu}\left(A^\nu + \Gamma^\nu\, \mathrm{id}\right)$$

$$\Gamma^\nu = g^{\mu\rho}\, \Gamma^\nu_{\mu\rho} \quad \text{Christoffel symbols of } g_{\mu\nu}$$

$$\Omega_{\mu\nu} = \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu + [\omega'_\mu, \omega'_\nu] \quad \text{curvature of } \nabla$$

Notation: $E_{;\mu}{}^{\mu} := \nabla_\mu \nabla^\mu E$

then

$$a_0(x,P) = (4\pi)^{-m/2} \underset{\text{rk } E}{\operatorname{Tr}}(id)$$

$$a_2(x,P) = (4\pi)^{-m/2} \operatorname{Tr}\left(-\frac{R}{6} id + E\right)$$

$$a_4(x,P) = (4\pi)^{-m/2} \frac{1}{360} \operatorname{Tr}\left(-12 R_{;\mu}{}^{\mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60 RE + 180 E^2 + 60 E_{;\mu}{}^{\mu} + 30\, \Omega_{\mu\nu}\Omega^{\mu\nu}\right)$$

Then use this to compute spectral action expansion in the following case:

$$\begin{cases} \mathcal{A} = C^\infty(M) \otimes M_N(\mathbb{C}) = C^\infty(M, M_N(\mathbb{C})) \\ \mathcal{H} = L^2(M,S) \otimes M_N(\mathbb{C}) \quad \text{inner prod: } \int_M \operatorname{Tr}(\xi(x)^* \eta(x)) \sqrt{g}\, d^4x = \langle \xi, \eta \rangle \\ D = \not{\partial}_M \otimes 1 \quad (D = 0 \text{ on } M_N(\mathbb{C}) \text{ part}) \\ \gamma = \gamma_5 \otimes 1 \end{cases}$$

$$\not{\partial}_M = \sqrt{-1}\, \gamma^\mu \nabla^S_\mu \qquad g_{\mu\nu} = e^a_\mu e^b_\mu \delta_{ab} \qquad \gamma_a = e^\mu_a \gamma_\mu$$

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab} \qquad \gamma_{ab} := \frac{1}{4}[\gamma_a, \gamma_b]$$

$$\gamma_a\gamma_b + \gamma_b\gamma_a \qquad [\gamma_{ab}, \gamma_{cd}] = \delta_{cb}\gamma_{ad} - \delta_{ca}\gamma_{bd} - \delta_{db}\gamma_{ac} + \delta_{da}\gamma_{bc}$$

$$\nabla^S_\mu = \partial_\mu + \underbrace{\frac{1}{2}\omega^{ab}_\mu \gamma_{ab}}_{\omega_\mu}$$

fluctuations of $\not{\partial}_M \otimes 1$: $\sum_j a_j [D, b_j]$

$A = A_\mu \gamma^\mu \quad A_\mu \in C^\infty(M, M_N(\mathbb{C})) \quad A_\mu = A_\mu^*$

$SU(N)$ gauge field w/ "unimodularity" $\operatorname{Tr}(A) = 0$

Can assume $Tr(A)=0$ since when taking

$$A + JAJ^* = \gamma^\mu \, ad(A_\mu)$$

eliminate the $U(1)$-part of A $\Rightarrow$ $SU(N)$ instead of $U(N)$-field

$(J_M \gamma^\mu J_M^{-1} = -\gamma^\mu ; \; Jm = m^*)$

$$D_A = D + A + JAJ^* = \sqrt{-1}\, e_a^\mu \gamma^a \left((\partial_\mu + \omega_\mu) 1 \otimes id_{M_N(\mathbb{C})} + 1 \otimes \mathbb{A}_\mu \right)$$

with $\mathbb{A}_\mu = \sqrt{-1}\, ad(A_\mu)$

Action for Dirac operator coupled to fermions

$$\langle \bar\psi , D_A \psi \rangle_{\mathcal{H}}$$

Yang-Mills $SU(N)$-action

$$S_{YM}(A) = \frac{1}{4} \int_M \mathcal{L}_{YM}(A) \, dv$$

$$\mathcal{L}_{YM}(A) = Tr\left(F_{\mu\nu} F^{\mu\nu} \right)$$

$F_{\mu\nu}$ = curvature of A_μ

Lagrangian of YM coupled to gravity:

$$Tr\left(f\left(\frac{D}{\Lambda}\right)\right) \sim \frac{1}{4\pi^2} \int_M \mathcal{L}(g,A) \sqrt{g} \, d^4x \quad \text{where}$$

$$\mathcal{L}(g,A) = 2N^2 \Lambda^4 f_4 + \frac{N^2}{6} \Lambda^2 f_2 R + \frac{f(0)}{6} \mathcal{L}_{YM}(A)$$
$$- \frac{N^2 f(0)}{80} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

Weyl curvature tensor

In fact get (from Gilkey's thm.) $\quad \dfrac{P = D^2}{s = -R}$

$$a_0(P) = \frac{N^2}{4\pi^2} \int_M \sqrt{g}\, d^4x$$

$$a_2(P) = -\frac{N^2}{48\pi^2} \int_M s\sqrt{g}\, d^4x = \frac{N^2}{48\pi^2}\int_M R\sqrt{g}\, d^4x$$

$$a_4(P) = \frac{1}{16\pi^2}\frac{N^2}{360}\int_M \left(5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\sqrt{g}\, d^4x$$

$$+ \frac{1}{24\pi^2}\int_M \mathrm{Tr}\left(F_{\mu\nu}\bar{F}^{\mu\nu}\right)\sqrt{g}\, d^4x$$

$N^2 = \dim M_N(\mathbb{C})$

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\nu|\sigma]} - g_{\nu[\rho}R_{\mu|\sigma]}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R$$

$$\frac{N^2}{16\pi^2}\int \left(-\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*\right)\sqrt{g}\, d^4x$$

Note: Gravity coupled to matter

Einstein equations for the gravitational field

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda_c g_{\mu\nu} = -8\pi G \, T_{\mu\nu}$$

energy momentum tensor

- metric tensor $g_{\mu\nu}$
- Ricci curvature $R_{\mu\nu}$
- Scalar curvature $s = -R$
- Λ_c = cosmological constant

Einstein-Hilbert action

$$S_{EH}(g) = \frac{1}{16\pi G}\left(\int_M R \, dv + 2\Lambda_c \int_M dv\right)$$

gives as variational equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda_c g_{\mu\nu} = 0$$

Einstein equations in vacuum

In the presence of matter:

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{SM}$$

gravity minimally coupled to matter

$$S = S_{EM} + S_{SM}$$

in curved field $g_{\mu\nu}$

Variation: $$\delta S_{SM} = \frac{1}{2} \int T^{\mu\nu} \delta g_{\mu\nu} \sqrt{g} \, d^4x$$

$$\delta S = \frac{1}{16\pi G} \int \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda_c g^{\mu\nu} + 8\pi G \, T^{\mu\nu}\right) \delta g_{\mu\nu} \sqrt{g} \, d^4x + \delta S_{SM}$$

variational equation becomes Einstein-Hilbert with energy-momentum tensor equal to that of δS_{SM}

($\delta S_{SM} = 0$ = energy momentum conservation)

Higher derivative terms in gravity

$S_{EH}(g) \rightsquigarrow$ gravity non-renormalizable

At low energy $\rightsquigarrow$ effective field theory

Adding higher derivative terms to S_{EH}
does not change low energy but
Corrects for non-renormalizability at high energies
(still other problems w/ unitarity)

invariant expressions that are quadratic in the curvature

$$\int_M \left(\frac{1}{2\eta} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - \frac{\omega}{3\eta} R^2 + \frac{\theta}{\eta} E \right) \sqrt{g}\, d^4x$$

Weyl curvature

E function of the curvature tensor such that

$$\chi(M) = \frac{1}{32\pi^2} \int_M E \sqrt{g}\, d^4x \qquad \text{Euler characteristic}$$

Sometimes written equivalently as

$$E = R^*R^* := R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$
$$= \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}{}_{\mu\nu} R^{\gamma\delta}{}_{\rho\sigma}$$

Gives the Pontryagin class

Fermionic part of the action

KO-dim $(M \times F) = 2 \bmod 8$

$J^2 = -1 \quad JD = DJ \quad J\gamma = -\gamma J$

$\mathcal{H}^+ =$ even part of $\mathcal{H} = \{\xi \in \mathcal{H} \mid \gamma\xi = \xi\}$

Antisymmetric bilinear form on $\mathcal{H}^+$:

$$\mathcal{A}_D(\xi, \xi') := \langle J\xi', D\xi \rangle \qquad \forall_0 \xi, \xi' \in \mathcal{H}^+$$

$$\langle J\xi, D\xi' \rangle = -\langle J\xi, J^2 D\xi' \rangle = -\langle JD\xi', \xi \rangle$$
$$= -\langle DJ\xi', \xi \rangle = -\langle J\xi', D\xi \rangle$$

using $J^2 = 1 \quad JD = DJ \quad D^* = D$
$\gamma JD = JD\gamma$

Also for $D_u = Ad(u)\, D\, Ad(u^*)$

with $Ad(u) = u(u^*)^0 = u J u J^{-1}$ have

$$\mathcal{A}_D(\xi', \xi) = \mathcal{A}_{D_u}(Ad(u)\xi', Ad(u)\xi)$$

<u>Pfaffian</u>: Antisymmetric bilinear form $\mathcal{A}$

$$Pf(\mathcal{A}) = \int e^{-\frac{1}{2}\mathcal{A}(\tilde{\xi})} \mathcal{D}[\tilde{\xi}]$$

where $\mathcal{H}^+_{cl} = \{\tilde{\xi} \mid \xi \in \mathcal{H}^+\}$, $\tilde{\xi} =$ Grassmann variable

satisfying $\tilde{\xi}_1\tilde{\xi}_2 + \tilde{\xi}_2\tilde{\xi}_1 = 0$ and $\int \tilde{\xi}\, d\tilde{\xi} = 1$

Simplest example:

$$\mathcal{a}(\xi',\xi) = a(\xi_1'\xi_2 - \xi_1^2\xi_1) \qquad \begin{matrix}\xi=(\xi_1,\xi_2)\\ \xi'=(\xi_1',\xi_2')\end{matrix}$$

$$\int e^{-\frac{1}{2}\mathcal{a}(\tilde\xi)} D[\tilde\xi] = \int e^{-a\tilde\xi_1\tilde\xi_2}\, d\tilde\xi_1\, d\tilde\xi_2 = a$$

(expanding exp: all other integrals vanish)

Notice: $\mathcal{a}$ = antisymmetric then
$\mathcal{a}(\xi,\xi)=0$ for ordinary (commuting) variables but
$\mathcal{a}(\tilde\xi,\tilde\xi)\neq 0$ for anticommuting case

Spectral action with fermionic terms for $(\mathcal{A},\mathcal{H},D,J,\gamma)$:

$$\mathrm{Tr}(f(D_A)) + \frac{1}{2}\langle J\tilde\xi, D_A\tilde\xi\rangle = S(A,\xi)$$

(D,J,γ) given and used all in def. of action
contain parameters

fields: bosons & fermions
$\tilde\xi\in\mathcal{H}^+$

Pfaffian is a "square root of a determinant"
$\mathcal{a} \leftrightarrow A$ = antisymm. matrix
$\mathrm{Pf}(\mathcal{a}) \leftrightarrow \det(A)^{1/2}$

So resolves the "fermion doubling problem"

Relations between the parameters

- Normalization of kinetic term for Higgs:

$H = \frac{\sqrt{a}\, f_0}{\pi}\ \varphi$ change of variable

gives $\int_M \frac{1}{2} |D_\mu H|^2 \sqrt{g}\, d^4x$

$\Rightarrow$ yields also normalization of all the Yang-Mills terms

$$-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

for all the gauge fields B_μ, W^a_μ, V^a_μ

$\Rightarrow$ unification of the coupling constants (as in GUT's)

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \qquad g_3^2 = g_2^2 = \frac{5}{3} g_1^2$$

Mass relation at unification: (specific prediction in this model)

Y_u, Y_d, Y_ν, Y_e enter in terms w/ coupling of Higgs field w/ fermions

in expressions $\qquad x = d, u, \nu, e$

dimensionless $\rightarrow$ $k_x = \frac{\pi}{\sqrt{a} f_0} Y_x$

where $a = \mathrm{Tr}\left(Y_\nu^* Y_\nu + Y_e^* Y_e + 3 (Y_u^* Y_u + Y_d^* Y_d) \right)$ dim of $(\text{mass})^2$

$$(k_u)_{\sigma k} = \frac{g}{2M} m^{\sigma}_{(u)} \delta^{k}_{\sigma}$$

$$(k_d)_{\sigma k} = \frac{g}{2M} m^{\mu}_{(d)} C_{\sigma\mu} \delta^{\rho}_{\mu} C^{*}_{\rho k}$$

$$(k_\nu)_{\sigma k} = \frac{g}{2M} m^{\sigma}_{(\nu)} \delta^{k}_{\sigma}$$

$$(k_e)_{\sigma k} = \frac{g}{2M} m^{\mu}_{(e)} C^{lep}_{\sigma\mu} \delta^{\rho}_{\mu} (C^{lep}_{\rho k})^{*}$$

from the identif. of terms in spectral action w/ terms in SM Lagrangian

$M = M_W$

Obtain then

$$\mathrm{Tr}\left(k_\nu^* k_\nu + k_e^* k_e + 3(k_u^* k_u + k_d^* k_d) \right) = 2g^2$$

using $k_x = \frac{\pi}{\sqrt{a}\, f_0} Y_x$ expression for a as function of the Y_x

and previous relation $\frac{g^2}{2\pi^2} f_0 = \frac{1}{4}$

to relate : $\frac{\pi^2}{f_0} = 2g^2$

$$\Downarrow$$

$$\sum_{\sigma} (m^{\sigma}_{\nu})^2 + (m^{\sigma}_{e})^2 + 3(m^{\sigma}_{u})^2 + 3(m^{\sigma}_{d})^2 = 8 M_W^2$$

over generations

This relation holds at unification scale : too high energy for a direct check

Running down to ordinary scales by RGE get compatible prediction for mass of top quark

factor of correction down to 170

(realistic when correction from seesaw mechanism for neutrino masses m_{ν_τ}-term taken into account) 173.68 GeV.

Higgs mass: get from spectral action a term

$$\frac{f_0 b}{2\pi^2}\int |\varphi|^4 \sqrt{g}\, d^4x = \frac{\pi^2}{2f_0}\frac{b}{a^2}\int |H|^4 \sqrt{g}\, d^4x$$

after same change of variables $H = \frac{\sqrt{a f_0}}{\pi}\varphi$

$\Rightarrow$ quartic self coupling of the Higgs

$\tilde{\lambda} = g_3^2 \frac{b}{a^2}$ (again using $\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4}$)

runs with energy scale according to RGE

$$m_H^2 = 8\lambda \frac{M^2}{g^2} \qquad m_H = \sqrt{2}\,\lambda \frac{2M}{g} \qquad \lambda = 4\tilde{\lambda} = 4 g_3^2 \frac{b}{a^2}$$

$\frac{2M}{g} = v =$ Higgs vacuum expectation

$H = \frac{\sqrt{a}}{\sqrt{2}g}(1+\psi)$

$= (\frac{2M}{g} + \mathcal{H} - i\phi^0, -i\sqrt{2}\phi)$

RGE <u>at one-loop</u>

$$\frac{d\lambda}{dt} = \lambda\gamma + \frac{1}{8\pi^2}(12\lambda^2 + B)$$

with

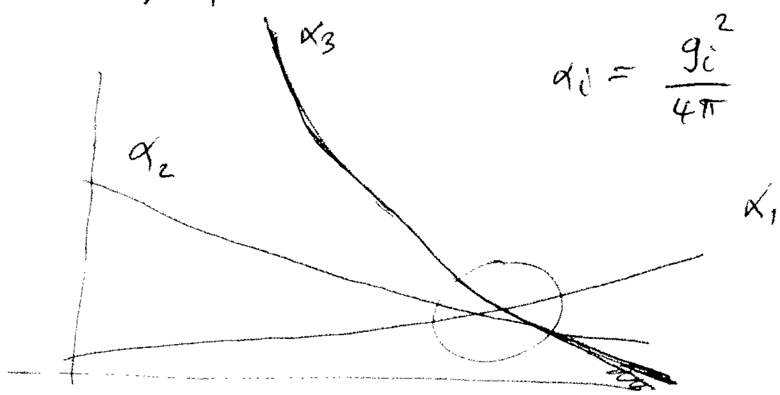
$$\gamma = \frac{1}{16\pi^2}(12 y_t^2 - 9 g_2^2 - 3 g_1^2)$$

$$B = \frac{3}{16}(3 g_2^4 + 2 g_1^2 g_2^2 + g_1^4) - 3 y_t^4$$

(Note: at one-loop RGE for coupling constants solved indep. from other Yukawa param.)

Running of coupling constants:

$$\alpha_i = \frac{g_i^2}{4\pi}$$



Again correct y_t term by adding correction from y_{ν_t} from seesaw mechanism; Before correction $m_H = 170$
after: $m_H = 168$ GeV

final list of parameters

$f_0 \leftrightarrow$ gauge coupling constants at unification

$f_2, f_4 \leftrightarrow$ gravitational parameters:
Newton constant, gravitational constant,
parameters of higher derivative
terms

Action
$S(\mathcal{A},\mathcal{H},D,J,\gamma) = \mathrm{Tr}(f(\frac{D_A}{\Lambda})) + \langle J\tilde{\xi}, D\tilde{\xi}\rangle$
function of fields (A,ξ) $\quad \tilde{\xi} \in \mathcal{H}^+$

Functional integral

$$\int e^{\frac{i}{\hbar} S(\mathcal{A},\mathcal{H},D,J,\gamma)(A,\xi)} \mathcal{D}[(A,\xi)]\, \mathcal{D}[(\mathcal{A},\mathcal{H},D,J,\gamma)]$$

Integration with constraints

which data $(\mathcal{A},\mathcal{H},D,J,\gamma)$ are
spectral triples of $M \times F$-like spaces

Characterization of manifolds
and constraints on spectral geometry