Calculus model for Feynman integrals

(from Zee "QFT in a nutshell")

$$Z(J) = \int_{\mathbb{R}} dx \; e^{-\frac{1}{2}m^2x^2 - \frac{\lambda}{4!}x^4 + Jx}$$

$\boxed{\lambda = 0}$ Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy = 2\pi \int_0^{+\infty} r\,dr\; e^{-\frac{1}{2}r^2}$$

$$= 2\pi \int_0^{\infty} e^{-w} dw = 2\pi \qquad \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = (2\pi)^{1/2}$$

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}ax^2} dx = \left(\frac{2\pi}{a}\right)^{1/2} \qquad (*)$$

Check:

$$\langle x^{2n} \rangle = \frac{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}ax^2} x^{2n} dx}{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}ax^2} dx} = \frac{1}{a^n} \overbrace{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}^{(2n-1)!!} \qquad (**)$$

hint: act repeatedly on $(*)$ by $-2\frac{d}{da}$

The combinatorial factor $(2n-1)!!$ counts all possible ways of connecting pairs of $2n$ pts

$\underbrace{x\,x \cdots x\,x}_{2n}$ example $x\,x\,x\,x\,x\,x = x^6$

"Wick contractions"

With "source term"

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}ax^2 + Jx} dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{J^2/2a} \qquad (***)$$

Complete the square!

$$-\frac{a}{2}x^2 + Jx = -\frac{a}{2}\left(x^2 - \frac{2Jx}{a}\right) = -\frac{a}{2}\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a}$$

then change coordinates in the integral

$$y = x + \frac{J}{a}$$

(Other way: take derivatives in J then set $J=0$)
gives (**) from (***)

$\boxed{\lambda \neq 0}$: expand the exponential (perturbative expansion)

$$Z(J) = \int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}m^2x^2 + Jx}\left(1 - \frac{\lambda}{4!}x^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 x^8 + \cdots\right)$$

Each term is an integral of the form

$$\int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}mx^2 + Jx} \, x^{4n} \qquad (*)$$

Notice: $\frac{d}{dJ} e^{-\frac{1}{2}mx^2 + Jx} = e^{-\frac{1}{2}mx^2 + Jx} \cdot x$

and $(*) = \left(\frac{d}{dJ}\right)^{4n} \int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}mx^2 + Jx}$

$$= \left(\frac{2\pi}{m^2}\right)^{1/2} \left(\frac{d}{dJ}\right)^{4n} e^{J^2/2m^2}$$

Get:

$$\exp\left(-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4\right)$$

$$Z(J) = \left(1 - \frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8 + \cdots\right) \int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}m^2x^2 + Jx}$$

$$= \left(\frac{2\pi}{m^2}\right)^{1/2} e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} e^{J^2/2m^2}$$

(forget for simplicity $(\frac{2\pi}{m^2})^{1/2}$ factor)

Examples: • term of order λ and J^4 comes from

– term order J^8 in $e^{J^2/2m^2} = (1 + \dots + \underbrace{\frac{1}{4!(2m^2)^4} J^8} + \dots)$

– term of order λ in $e^{-\frac{\lambda}{4!}(\frac{d}{dJ})^4} = (1 + \underbrace{\frac{-\lambda}{4!}(\frac{d}{dJ})^4} + \dots)$

$$\frac{-\lambda}{4!}\left(\frac{d}{dJ}\right)^4 \left(\frac{J^8}{4!(2m^2)^4}\right) = \underline{\frac{8!\,(-\lambda)\,J^4}{(4!)^3(2m^2)^4}}$$

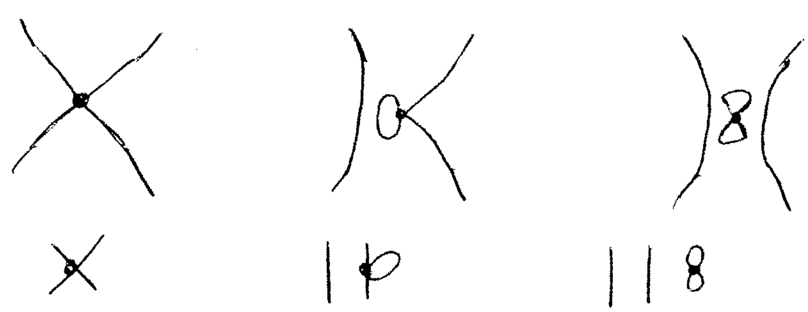
• term order λ^2, J^6:

$$\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8\left(\frac{J^{14}}{7!(2m^2)^7}\right) = \frac{14!(-\lambda)^2\ J^6}{(4!)^2 6!\,7!\,2(2m^2)^7}$$

• term order λ^2, J^4 $\quad \frac{12!(-\lambda)^2\ J^4}{(4!)^3 3!(2m^2)^6}$

• term order λ, J^0 is $\frac{\cancel{8}\ -\lambda}{2(2m^2)^2}$

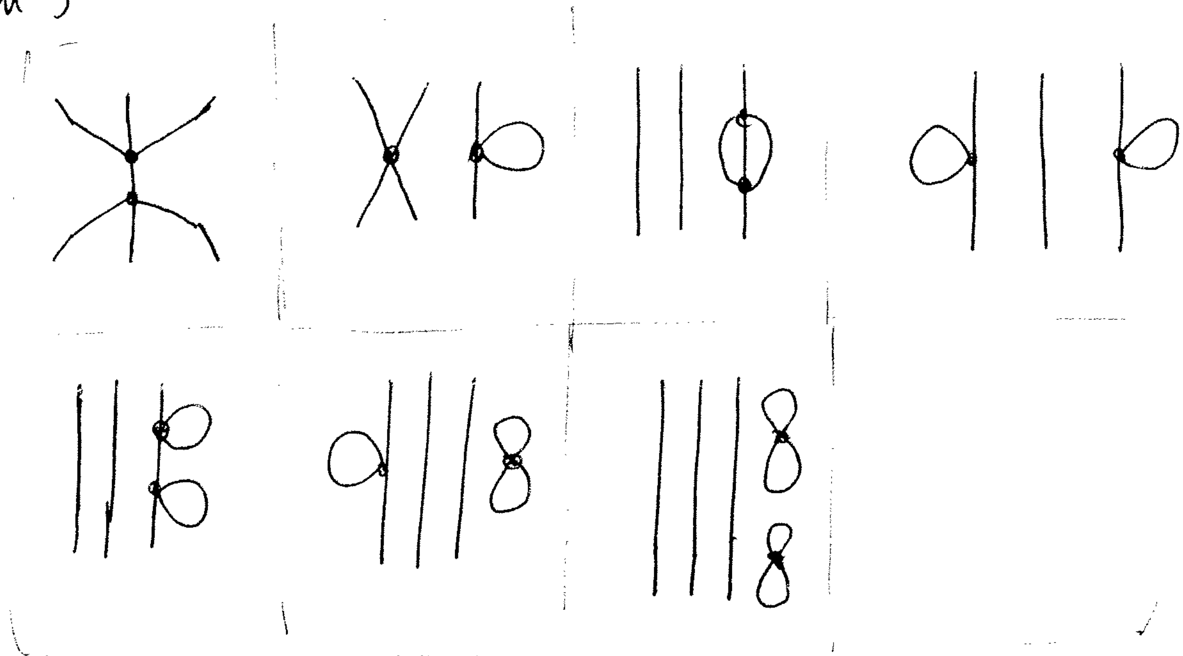
$\frac{1}{(m^2)^4}\lambda, J^4$:



each vertex : λ
each line : $\frac{1}{m^2}$

each external end : J

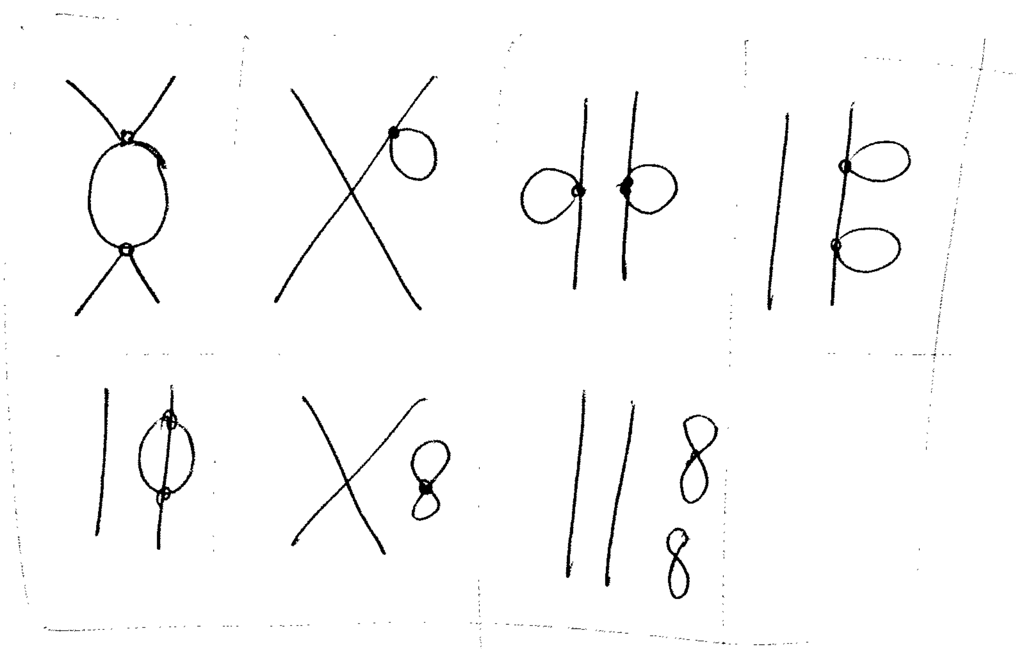
$\frac{1}{(m^2)^7} \lambda^2 J^6$:



Corresponding term in the expansion is:

$$\sum_{\gamma = \text{graph}} \frac{1}{\#\mathrm{Aut}(\gamma)} \frac{\lambda^{\#\text{vertices}} J^{\#\text{ext edges}}}{(m^2)^{\#\text{int edges}}}$$

$(\frac{1}{m^2})^6 \lambda^2 J^4$:



- $Z(J)$ as generating function: Green functions

$$Z(J) = \sum_{k=0}^{\infty} \frac{1}{k!} J^k \int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}m^2x^2 - \frac{\lambda}{4!}x^4} x^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} J^k G_k \quad \text{Green function } (k\text{-point})$$

$$G_k = \int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}m^2x^2 - \frac{\lambda}{4!}x^4} x^k$$

- Passing to connected graphs: "change of variables"

$$Z(\lambda, J) = Z(\lambda, J=0) \cdot e^{W(J,\lambda)}$$

$$= Z(\lambda, J=0) \cdot \sum_{N=0}^{\infty} \frac{1}{N!} W(J,\lambda)^N$$

– One variable to many (finitely many) variables:

A = real $N \times N$ symmetric ($A^T = A$) matrix
invertible $\det A \neq 0$

$A = (A_{ij})_{i=1,\dots,N;\ j=1,\dots,N}$ $\qquad x = (x_j)_{j=1,\dots,N}$

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1 \dots dx_N \; e^{-\frac{1}{2} x^T A x + Jx} = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} \; e^{\frac{1}{2} J A^{-1} J}$$

Check: Use an orthogonal matrix O to diagonalize A and then use 1-dimensional result to compute

Wick contractions:

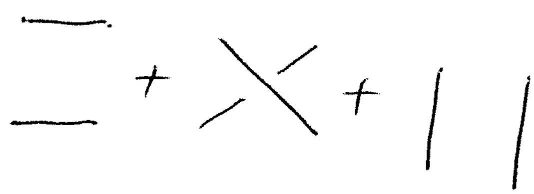
$$\langle x_{i_1} x_{j_1} \cdots x_{i_\ell} x_{j_\ell} \rangle = \frac{\int_{\mathbb{R}^N} dx_1 \cdots dx_N \; e^{-\frac{1}{2} x^T A x} \, x_{i_1} x_{j_1} \cdots x_{i_\ell} x_{j_\ell}}{\int_{\mathbb{R}^N} dx_1 \cdots dx_N \; e^{-\frac{1}{2} x^T A x}}$$

$$= \sum_{\substack{\text{Wick} \\ \text{contractions}}} (A^{-1})_{a_1 b_1} \cdots (A^{-1})_{a_\ell b_\ell}$$

(ways of pairing the $x_{i_1} \cdots x_{j_\ell}$)

Example:

$$\langle x_i x_j x_k x_\ell \rangle = A^{-1}_{ij} A^{-1}_{k\ell} + A^{-1}_{i\ell} A^{-1}_{jk} + A^{-1}_{ik} A^{-1}_{j\ell}$$



With interaction term and source term:

$$Z(J) = \int_{\mathbb{R}^N} dx_1 \cdots dx_N \; e^{-\frac{1}{2} x^T A x - \frac{\lambda}{4!} x^4 + J \cdot x}$$

$$\left(x^4 := \sum_{i=1}^N x_i^4 \right)$$

Get:

$$\boxed{Z(J) = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} \; e^{-\frac{\lambda}{4!} \sum_{i=1}^N \left(\frac{\partial}{\partial J_i}\right)^4} \; e^{\frac{1}{2} J A^{-1} J}}$$

Field theory

- Dimension D ("same" theory can be looked at in different dimensions)

- Lagrangian density

  functional on classical fields $\phi \in C^\infty(\overset{\mathbb{R}^D}{\overset{\shortparallel}{M}}, \mathbb{C})$

  or sections of some bundle over M

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Lorentzian signature

$$(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$$

$D=4$: $g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ $\qquad (\partial\phi)^2 = (\partial_0\phi)^2 - \sum_{i=1}^{3}(\partial_i\phi)^2$

Free field part: $\mathcal{L}_0(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$

Interaction part: $\mathcal{L}_{int}(\phi)$ = Polynomial in ϕ deg > 2

Physical theory $\mathcal{T} = \{\mathcal{L}(\phi), D\}$

e.g. $\mathcal{T} = \phi^3_6$ means $\mathcal{L}_{int}(\phi) = \frac{\lambda}{3!}\phi^3$ and $D = 6$

Expectation values

Observables: functions $\mathcal{O}$ of the classical fields

$\mathcal{O}(\phi)$ $\quad\quad$ $\mathcal{O}: C^\infty(M, \mathbb{C}) \longrightarrow \mathbb{C}$

~~possibly~~ generally non-linear functional
w/ some assumption of regularity
(e.g. continuous, functional derivatives, ...)

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}(\phi)\, e^{i\frac{S(\phi)}{\hbar}}\, D[\phi]}{\int e^{i\frac{S(\phi)}{\hbar}}\, D[\phi]} \qquad (*)$$

Probability amplitude: $\quad$ classical action functional

$$S(\phi) = \int_{\mathbb{R}^D} \mathcal{L}(\phi)\, d^D x$$

$$e^{\frac{i S(\phi)}{\hbar}}$$

$\hbar = \frac{h}{2\pi}$ Planck constant

Replaces classical (real valued) probability by complex valued probability amplitude

Notice: Important problem with $(*)$:

the measure $D[\phi]$ is ill defined mathematically in general

Green's functions:

$$G_N(x_1,\ldots,x_N) = \frac{\int e^{\frac{iS(\phi)}{\hbar}} \phi(x_1)\ldots\phi(x_N)\, D[\phi]}{\int e^{\frac{iS(\phi)}{\hbar}} D[\phi]}$$

Gaussian measure:

$$S(\phi) = S_0(\phi) + S_{int}(\phi) \quad \text{corresponding to}$$

$$\mathcal{L}(\phi) = \mathcal{L}_{free}(\phi) + \mathcal{L}_{int}(\phi)$$

$$S_0(\phi) = \int \mathcal{L}_{free}(\phi)\, d^Dx = \int \left(\tfrac{1}{2}(\partial\phi)^2 - \tfrac{m^2}{2}\phi^2\right) d^Dx$$

$$S_{int}(\phi) = \int \mathcal{L}_{int}(\phi)\, d^Dx$$

$$\boxed{d\Lambda = e^{\frac{iS_0(\phi)}{\hbar}} D[\phi]} \qquad \text{"imaginary Gaussian measure"}$$

Expand ~~the~~ $S_{int}(\phi)$ exponential:

$$G_N(x_1,\ldots,x_N) = \frac{\sum_{k=1}^{\infty} \frac{i^k}{k!} \int \phi(x_1)\ldots\phi(x_N)\, S_{int}(\phi)^k\, d\Lambda[\phi]}{\sum_{k=1}^{\infty} \frac{i^k}{k!} \int S_{int}(\phi)^k\, d\Lambda[\phi]}$$

Now they look like integrals of polynomials under a Gaussian

Wick rotation : Euclidean signature

$$\mathcal{L}_{Eucl}(\phi_{Eucl}) = \frac{1}{2}(\partial\phi_{Eucl})^2 + \frac{m^2}{2}\phi_{Eucl}^2 + \mathcal{L}_{int}(\phi_{Eucl})$$

Now Euclidean metric $g^{\mu\nu} = \delta^{\mu\nu}$

$$(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \sum_{\mu=1}^{4}(\partial_\mu\phi)^2$$

One passes from $\mathcal{L}(\phi)$ to $\mathcal{L}_{Eucl}(\phi_{Eucl})$
by effect of changing $\boxed{t \longmapsto it}$ Wick rotation
so that $\int \phi_E(x_1)\cdots\phi_E(x_N)\, e^{-\frac{S_E(\phi_E)}{\hbar}} D[\phi_E]$
instead of $\int \phi(x_1)\cdots\phi(x_N)\, e^{\frac{i S(\phi)}{\hbar}} D[\phi]$

$$S_E(\phi_E) = \int \mathcal{L}_E(\phi_E)\, d^Dx$$

$$\langle \mathcal{O} \rangle_E = \frac{\int \mathcal{O}(\phi_E)\, e^{-\frac{S_E(\phi_E)}{\hbar}} D[\phi_E]}{\int e^{-\frac{S_E(\phi_E)}{\hbar}} D[\phi_E]}$$

(drop subscripts Eucl for simplicity)

- Integration with Gaussian measure :
  as in finite dimensional model
  (integration by parts)

Schwinger functions $S_N(x_1,\dots,x_N) = \dfrac{\int \phi(x_1)\cdots\phi(x_N)\, e^{-\frac{S(\phi)}{\hbar}} D[\phi]}{\int e^{-\frac{S(\phi)}{\hbar}} D[\phi]}$

Integration by parts :

V = vector space V^* = dual = $\mathrm{Hom}(V, k)$

$Q \in V^* \otimes V^*$ non-degenerate quadratic form

inverse : Q^{-1}

$$Q \in \mathrm{Hom}(V, V^*) \simeq V^* \otimes V^*$$
$$Q^{-1} \in \mathrm{Hom}(V^*, V) \simeq V^* \otimes V^*$$

Q invertible, symmetric

Given $L \in V^* = \mathrm{Hom}(V, k)$

$$\boxed{\partial_{Q^{-1}(L)} \tfrac{1}{2} Q = L}$$

$$\int P(X)\, L(X)\, \exp\left(-\frac{Q(X)}{2}\right) D[X] =$$

$$= -\int P(X)\, \partial_{Q^{-1}(L)} \left(\exp\left(-\frac{Q(X)}{2}\right)\right) D[X]$$

$$= \int \partial_{Q^{-1}(L)} (P(X))\, \exp\left(-\frac{Q(X)}{2}\right) D[X].$$

Formalizes the computation of the integrals of the form

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2} x^T A x}\, x_{i_1} \cdots x_{i_\ell}\, dx_1 \cdots dx_N$$

seen at the beginning, using integrations by parts

$\rightsquigarrow$ Generates a sum of terms that can be parameterized by graphs

Feynman rules : (case of scalar field theory)

$$S_N(x_1,\dots,x_N) = \sum_{\Gamma} \int \frac{V(\Gamma)(p_1,\dots,p_N)}{\# \mathrm{Aut}(\Gamma)} \cdot e^{i(x_1 p_1 + \dots + x_N p_N)} \cdots \frac{dp_1}{(2\pi)^D} \cdots \frac{dp_N}{(2\pi)^D}$$

Γ = Feynman graph of the theory

$\Gamma^{(0)}$ = vertices

$\Gamma^{(1)}$ = edges (oriented)

N external edges

$\partial_j \quad j \in \{0,1\} \qquad \partial_j : \Gamma^{(1)} \to \Gamma^{(0)} \cup \{1,2,\dots,N\}$

I = collection of all the monomials that appear in the Lagrangian $\mathcal{L}_{int}(\phi)$

$i : \Gamma^{(0)} \longrightarrow I$ (each vertex $\rightsquigarrow$ a monomial)

so that degree $(i(v))$ = valence (v)

Rules :

- Each external line $\rightsquigarrow$ propagator $\dfrac{1}{p_i^2 + m^2}$
- Each internal line $l \mapsto$ a momentum variable k_l and a propagator $\dfrac{1}{k^2+m^2} \dfrac{d^D k}{(2\pi)^D}$
- Each vertex with $i(v) = \dfrac{-\lambda}{d!}\phi^d$ momentum conservation
  $\lambda (2\pi)^D \delta \left(\sum_{\partial_0(l)=v} k_l - \sum_{\partial_1(l)=v} k_l \right)$
- vertex w/ $i(v) = -\frac{w}{2}(\partial\phi)^2 \rightsquigarrow w(2\pi)^D k^2 \delta\left(\sum k_l - \sum_{\partial_1(l)=v} k_l\right)$