Examples of coproduct calculations in the Connes-Kreimer Hopf algebra.

**Primitive element**

\[ \Delta(-\textcircled{1}) = -\textcircled{1} \otimes 1 + 1 \otimes -\textcircled{1} \]

**Non-primitive**:

\[ \Delta(-\textcircled{1}) = -\textcircled{1} \otimes 1 + 1 \otimes -\textcircled{1} + \sum_{i=0}^{\infty} (-\textcircled{i}) \otimes (-\textcircled{1}) \]

Two possible valence two vertices

Two different \( \chi(\mathbf{x}) \) in monomials of Laprangian

\[ \Delta(-\textcircled{1}) = -\textcircled{1} \otimes 1 + 1 \otimes -\textcircled{1} + 2 \otimes -\textcircled{1} \]

\[ \Delta(-\textcircled{1}) = -\textcircled{1} \otimes 1 + 1 \otimes -\textcircled{1} + 2 \otimes -\textcircled{1} + 2 \otimes -\textcircled{1} + \]

\[ \uparrow \text{quadrate here} \]

\[ \downarrow \text{linear here} \]
Birkhoff factorization of loops

\[ \Delta \subset \mathbb{C} \text{ small disk centered at } z = 0 \]
\[ C = \exists \Delta \text{ circle around } z = 0 \]
\[ C_+ = \text{two connected components of } \mathbb{P}^1(C) \setminus C \]
\[ 0 \in C_+ \quad \infty \in C_- \]

\[ G(C) = \text{connected complex Lie group} \]

Loop: smooth map \( \gamma : C \to G(C) \)

\( \gamma \) admits a Birkhoff factorization iff

\[ \exists \gamma_+, \gamma_- \text{ holomorphic functions} \]

\[ \gamma_+ : C_+ \to G(C) \quad \text{extend to} \quad \gamma_+ \text{ (cont) functions on } \mathbb{C}_+ = \mathbb{C} \]

\[ \gamma_- : C_- \to G(C) \quad \text{extend to} \quad \gamma_- \text{ (cont) functions on } \mathbb{C}_- = \mathbb{C} \]

\[ \gamma(z) = \gamma_-(z)^{-1} \gamma_+(z) \quad \text{for } z \in \mathbb{C} \]

\[ \bigotimes \]

Problem: When does \( \gamma \) have Birkhoff factorization?

\[ \bigotimes \text{ In general not all loops have } \bigotimes \]

Example: \( G(C) = GL_n(C) \)
Birkhoff factorization in $\mathrm{GL}_n(C)$: holomorphic bundles on the sphere

Grothendieck decomposition (classification of holomorphic vector bundles on the sphere $\mathbb{P}^1(C)$)

\[ E = L_1 \oplus \ldots \oplus L_n \quad \text{sum of line bundles} \]

\[ c_1(L_i) = k_i \in \mathbb{Z} \quad \text{Chern classes} \]

Given a holomorphic vector bundle $E$ on $\mathbb{P}^1(C)$
restrictions $E|_{\mathbb{C}^+}$ and $E|_{\mathbb{C}^-}$ can be trivialized
i.e. exist $\gamma_+: \mathbb{C}^+ \to \mathrm{GL}_n(C)$ holomorphic
local frames for trivialization of $E|_{\mathbb{C}^+}$

\[ E = E|_{\mathbb{C}^+} \cup E|_{\mathbb{C}^-} \quad \text{glued together along } C = \mathbb{C}^+ \cap \mathbb{C}^- \]

using a transition function
\[ \lambda: C \to \mathrm{GL}_n(C) \]

By (**), this transition function is of the form
\[ \lambda(z) = \begin{pmatrix} 2^{k_1} & 0 \\ 0 & 2^{k_n} \end{pmatrix} \]

Since $\lambda(z) = 2^{k_i}$ is the transition function for a line bundle $L_i$ with $c_1(L_i) = k_i$

Conclusion: $\gamma(z) = \gamma_-(z)^{-1} \lambda(z) \gamma_+(z)$ for $\mathrm{GL}_n(C)$
Question: are there groups $G(C)$ for which 
$\gamma(z) = \gamma_-(z)^{-1} \gamma_+(z)$ for any loop $\gamma$?

Connes-Kreimer: yes for pronipotent groups 
with a recursive formula for the factorization

Pronipotent $G = \varinjlim_n G_n$ affine group scheme

$G_n$ unipotent algebraic group

(e.g. upper triangular matrices)

Notice: Commutative Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ (graded) 
with $\mathcal{H}_0 = k$ (connected) 
$\iff$ pronipotent affine group scheme $G$ 

$G^* = G \times G_m$

Action of multiplicative group $G_m$ on $\mathcal{H}$ by 
$u^y(x) = u^y x$ if $x \in \mathcal{H}_n$

($y(x) =$ generator of grading)

$\bullet$ In terms of Lie algebra: extra generator $Z$

$[Z, X] = \gamma(X) \quad \forall X \in \text{Lie}(G)$
Translating from loops to Hopf algebra and homomorphisms

\[ \gamma : \Delta^* \rightarrow G(C) \quad \text{loop} \]

\[ \gamma(z) = \gamma_-(z)^{-1} \gamma_+(z) \quad \text{Birkhoff factorization} \]

\[ \gamma_{-} : P(C) \setminus \{0\} \rightarrow G(C) \quad \gamma_-(\infty) = 1 \]

\[ \gamma_+ : \Delta \rightarrow G(C) \]

\[ \phi \in \text{Hom}_A(H, K) \]

\[ \gamma : \Delta^* \rightarrow G(C) \]

\[ \text{Note: } G(K) = \text{Hom}_A(H, K) \]

\[ G(C) = \text{Hom}_A(H, C) \]

\[ \gamma : \Delta^* \rightarrow G(C) \]

where \( K = C[z, [z^{-1}]] = \mathbb{C}(\{z\}) \)
field of convergent Laurent series
= germs of meromorphic functions at \( z = 0 \)
$\mathcal{O} = \mathbb{C}[z^{-1}]$ \hspace{1cm} convergent power series \hspace{1cm} subring of $K = \mathbb{C}(z)$

$Q = \mathbb{C}[z^{-1}]$ \hspace{1cm} divergent part (pole part)

$\hat{Q} = \mathbb{C}[z^{-1}]$ \hspace{1cm} Laurent polynomial \hspace{1cm} regular at $\infty$

$\Phi_+ \in G(\mathcal{O}) = \text{Hom}_A(\mathcal{O}, \mathcal{O}) \iff \gamma_+ : \Delta \to G(C)$

$\Phi_- \in G(\hat{Q}) = \text{Hom}_A(\hat{Q}, \hat{Q}) \iff \gamma_- : \text{Pic}(\mathcal{O})_{\text{reg}} \to G(C)$

$\varepsilon : \hat{Q} \to \mathbb{C}$ \hspace{1cm} augmentation

$\varepsilon \circ \Phi_- = \varepsilon \iff \gamma_-(\infty) = 1$

Birkhoff factorization property: given

$\Phi \in \text{Hom}_A(\mathcal{O}, K) \equiv G(K)$

$\exists \Phi_+ \in G(\mathcal{O})$, $\Phi_- \in G(\hat{Q}) \wedge \varepsilon \circ \Phi_- = \varepsilon$

such that

$\Phi = (\Phi_- \circ S) \ast \Phi_+$
Thus (Comtet-Korepin)
\[ H = \oplus_{n \geq 0} V_n \quad H_0 = \mathbb{C} \]
For all \( \phi \in G(K) = \text{Hom}_A(H,K) \) \( \exists \phi_+, \phi_- \)
\[ \phi = (\phi_+ \circ S) \ast \phi_+ \]
\( \phi_+, \phi_- \) given by the recursive formula

\[
\begin{align*}
\phi_-(x) &= -T(\phi(x) + \sum \phi_-(x') \phi(x'')) \\
\phi_+(x) &= \phi(x) + \phi_-(x) + \sum \phi_-(x') \phi(x'')
\end{align*}
\]
where \( \Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x'' \)

Proof: 1) first need to show that \( \phi_- \) defined as \( \hat{\omega}(*) \)
is an algebra homomorphism
\( \phi_- \in \text{Hom}_A(H,K) \)

i.e. that \( \phi_-(xy) = \phi_-(x) \phi_-(y) \)

This depends crucially on the fact that projection onto polar part of a Laurent series
is a Rota-Baxter operator i.e.
\[ T(f)T(h) = -T(fh) + T(T(f)h) + T(fT(h)) \]
(i.e. behaves somewhat like integration by parts)
Notice then that in the formula

\[ \phi_+(x) = \phi(x) + \phi_-(x) + \Sigma \phi_-(x') \phi_+(x'') \]

the right hand side is

\[ \langle \phi_-, \Delta(x) \rangle = (\phi_+ \ast \phi)(x) \]

So \( \phi_+ = \phi_+ \ast \phi \) hence

\[ \phi = \phi_+ \ast \phi_+ = (\phi_+ \circ \mathcal{S}) \ast \phi_+ \]

\[ \Rightarrow \quad \text{BPHZ} = \text{Birkhoff factorization} \]

Take \( \phi = U \) i.e. on generators

\[ \chi = \Gamma \] of \( \mathcal{H} \) \( \ast \phi(\Gamma) = U(\Gamma) \)

unrenormalized

Feynman integral

(as Laurent series)

then formulae (*) for Birkhoff factorization

same as BPHZ formulae:

\[ \phi_-(x) = -T(\phi(x) + \Sigma \phi_-(x') \phi(x'')) \]

\[ \phi_+(x) = \phi(x) + \phi_-(x) + \Sigma \phi_-(x') \phi(x'') \]

\[ C(\Gamma) = -T(U(\Gamma) + \Sigma c(x) U(x')) \]

\[ R(\Gamma) = R(\Gamma) + C(\Gamma) \]

with \( \phi_- = C \) \quad \phi_+ = R
Renormalization group

Grading $\oplus_{n \geq 0} H_n$ on the Hopf algebra $H = H(T)$ of Feynman graphs gives 1-parameter family of automorphisms

$$\theta_t \in \text{Aut}(G(T)) \quad \forall t \in G_a(C) = \mathbb{C}$$

$$\frac{d}{dt} \theta_t \big|_{t=0} = \gamma \quad \text{grading operator} \quad \gamma(x) = n x \quad x \in H_n$$

$$\theta_t(x) = \exp(nt) \cdot x \quad \forall x \in H_n(T)$$

Action on loops $\gamma(z)$

Notice: energy scale dependence

$$\gamma(z) \quad \longleftrightarrow \quad \phi \in \text{Hom}(H,K)$$

$$\phi(\Gamma)(z) = U^2(\Gamma)$$

$$U^2(\Gamma, p_1, \ldots, p_n) = \int d^{D-2}k_1 \ldots d^{D-2}k_n \mu^2 L \quad \Gamma \quad p_1, \ldots, p_n, k_1, \ldots, k_n$$

Energy scale

So $\gamma(z) = \gamma^*_\mu(z)$
Proposition \( (\text{10}) \)

\[
(1) \quad \Theta_{\tau_{\mu}}(\gamma_{\mu}(z)) = \gamma_{\epsilon_{\mu}}(z)
\]

Action of grading \( \rightarrow \) scaling of mass parameter

pf: check on generators \( \Gamma \):

\[
\Theta_{\tau_{\mu}}(\gamma_{\mu}(z)) (\Gamma) = \exp(b_{\mu}(\Gamma) \tau_{\mu}) \gamma_{\mu}(\Gamma)
\]

\[
= \exp(b_{\mu}(\Gamma) \tau_{\mu}) \mu^{2} b_{\mu}(\Gamma) \int d^{D_{1}} k_{1} d^{D_{2}} k_{2} I_{\mu}(p_{1}, k_{1}, k_{2})
\]

\[
= (e^{\mu})^{b_{\mu}(\Gamma)} \int d^{D_{1}} k_{1} d^{D_{2}} k_{2} I_{\mu}(p_{1}, k_{1}) = \gamma_{\epsilon_{\mu}}(z)(\Gamma)
\]

(2) \[
\frac{\partial}{\partial \mu} \gamma_{\mu}(z) = 0 \quad \text{in the Birkhoff factorization}
\]

\[
\gamma_{\mu}(z) = \gamma_{\mu}(z)^{-1} \gamma_{\mu}(\tau_{\mu})
\]

pf: Counterterms \( C(\Gamma) = \phi(\Gamma) \)

- depend polynomially on \( p^{2} \) (external momenta)
  (recall discussion on BPHZ and local terms)
- Dimensional analysis \( \rightarrow \) depend polynomially on mass parameters \( m \)
  (for DimReg+MS not e.g. for on-shell)
- Laurent series expansion of loop \( \gamma_{\mu}(z) \): \( \mu^{2} \) dependence
  on \( \mu \) no log \( \mu \) powers
- Dimensional reasons \( \log(\frac{\mu^{2}}{\Lambda^{2}}) \) or \( \log(\frac{\mu^{2}}{m^{2}}) \) \( \rightarrow \) dependence
Summarize: Data of perturbative renormalization in CK theory

- Given $T$ (lagrangian etc.)
  $H(T)$ Hopf algebra of Feynman graphs
- Dual to affine group scheme $G_T$
- Unrenormalized values of Feynman graphs $U_\mu^2(p)$ define loop $\gamma_\mu(2)$ in $G_T(C)$
- BPHZ is Birkhoff factorization
  $\gamma_\mu(2) = \gamma_-(2) \gamma_\mu^+(2)$

$\gamma_-$ = counter terms $\gamma_\mu^+(0) =$ renormalized values of all Feynman graphs

- Scaling property
  $\Theta_{\epsilon_2}(\gamma_\mu(2)) = \gamma_{\epsilon_\mu}(2)$

$\frac{\partial}{\partial \mu} \gamma_-(2) = 0$

- Renormalization group action
  $\mu \frac{\partial}{\partial \mu}$ on $\gamma_{\mu^+}(0)$
Better description of Renormalization group

Thus:

- \( \gamma(z) \quad \theta_t z (\gamma(z)^{-1}) \) regular at \( z = 0 \)
- \( F_t := \lim_{z \to 0} \gamma(z) \theta_t z (\gamma(z)^{-1}) \)
  - is \( t \)-parameter subgroup of \( G_f(C) \)
  - i.e. \( F_t F_s = F_{t+s} \)
- \( F_t(x) \) polynomial in \( t \) for all \( x \in H \)
  - \( F_t \in \text{Hom}_A(H, C) \)
- \( F_t \gamma_{\mu+}(0) = \gamma_{e^{\mu} +}(0) \)
- \( \text{Res}_{z=0} (\gamma_{\mu}(z)) := -\left( \frac{d}{d\mu} \gamma_{\frac{\mu}{\mu}} \right)_{\mu=0} \)
  - \( \beta := \gamma \text{ Res}(\gamma) \)
  - Beta function \( \beta \in \text{Lie}(G_f) \)
  - then
    \[
    \frac{d}{dt} F_t \bigg|_{t=0} = \beta
    \]
  - infinitesimal generator of the renormalization group