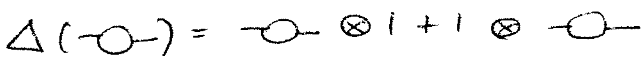
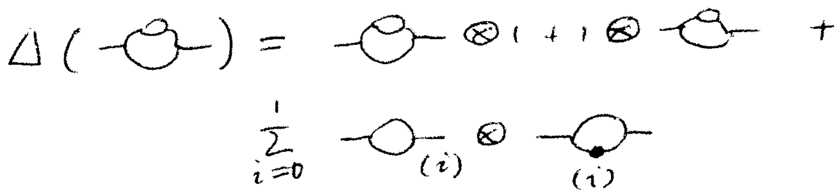
Examples of coproduct calculations in the Connes-Kreimer Hopf algebra

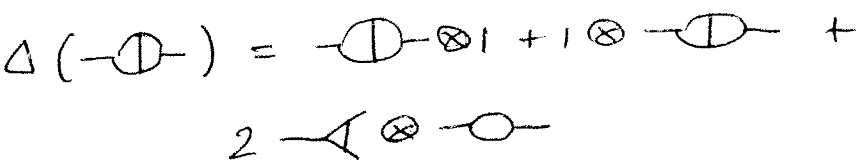
Primitive element 

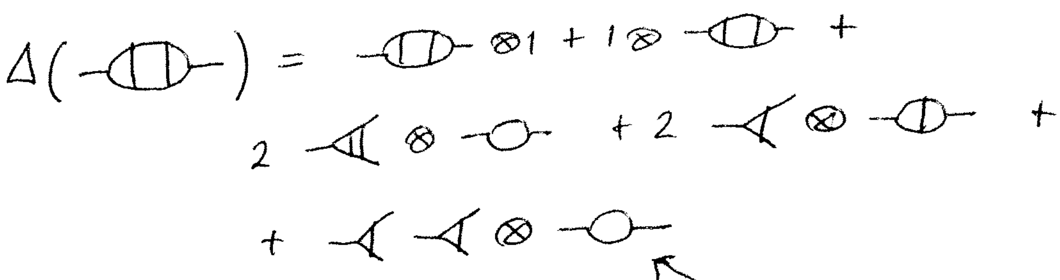


Non-primitive:



Two possible valence two vertices
two different $\chi(\gamma) \in$ monomials of Lagrangian





quadratic here
linear here

Birkhoff factorization of loops

$\Delta \subset \mathbb{C}$ small disk centered at $z=0$

$C = \partial\Delta$ circle around $z=0$

$C_\pm$ = two connected components of $\mathbb{P}^1(\mathbb{C}) \setminus C$

$0 \in C_+ \qquad \infty \in C_-$

$G(\mathbb{C})$ = connected complex Lie group

Loop : smooth map $\gamma : C \to G(\mathbb{C})$

γ admits a Birkhoff factorization iff
$\exists\ \gamma_+, \gamma_-$ holomorphic functions

$$\gamma_\pm : C_\pm \longrightarrow G(\mathbb{C})$$

extend to (cont) functions on $\partial C_+ = \partial C = C$

with $\gamma_-(\infty) = 1$ such that

$$\circledast \qquad \gamma(z) = \gamma_-(z)^{-1}\, \gamma_+(z) \qquad \text{for } z \in C$$

Problem : When does γ have Birkhoff factorization?

- In general not all loops have $\circledast$

Example : $G(\mathbb{C}) = GL_n(\mathbb{C})$

Birkhoff factorization in $Gl_n(\mathbb{C})$: holomorphic bundles on the sphere

Grothendieck decomposition (classification of holomorphic vector bundles on the sphere $\mathbb{P}^1(\mathbb{C})$)

$$(**) \quad E = L_1 \oplus \cdots \oplus L_n \qquad \text{sum of line bundles}$$

$c_1(L_i) = k_i \in \mathbb{Z}$ Chern classes

Given a holomorphic vector bundle E on $\mathbb{P}^1(\mathbb{C})$

restrictions $E|_{C_+}$ and $E|_{C_-}$ can be trivialized

i.e. exist $\gamma_\pm : C_\pm \to Gl_n(\mathbb{C})$ holomorphic

local frames for trivialization of $E|_{C_\pm}$

$E = E|_{C_+} \cup_\lambda E|_{C_-}$ glued together along $C = \partial C_+ = \partial C_-$ using a transition function

$$\lambda : C \to Gl_n(\mathbb{C})$$

By $(**)$ this transition function is of the form

$$\lambda(z) = \begin{pmatrix} z^{k_1} & & 0 \\ & \ddots & \\ 0 & & z^{k_n} \end{pmatrix}$$

Since $\lambda_i(z) = z^{k_i}$ is the transition function for a line bundle L_i with $c_1(L_i) = k_i$

Conclusion: $\quad \gamma(z) = \gamma_-(z)^{-1} \lambda(z) \gamma_+(z) \quad$ for $Gl_n(\mathbb{C})$

Question: are there groups $G(\mathbb{C})$ for which
$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)$ for any loop γ?

Connes-Kreimer: Yes for prounipotent groups
with a recursive formula for the factorization

Prounipotent $\quad G = \varprojlim_n G_n \quad$ affine group scheme

G_n unipotent algebraic group

(e.g. upper triangular matrices)

Notice: Commutative Hopf algebra $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ (positively graded)
with $\mathcal{H}_0 = k$ (connected)
$\Longleftrightarrow$ pro-unipotent affine group scheme G

$$G^* = G \rtimes \mathbb{G}_m$$

Action of multiplicative group $\mathbb{G}_m$ on $\mathcal{H}$ by
$u^Y(X) = u^n X \quad$ if $X \in \mathcal{H}_n$

($Y(X) = nX$ generator of grading)

- In terms of Lie algebra: extra generator Z
$[Z, X] = Y(X) \qquad \forall X \in Lie(G)$

Translating from Loops to Hopf algebra and homomorphisms

$\gamma : \Delta^* \longrightarrow G(\mathbb{C})$ loop

$\gamma(z) = \gamma_-(z)^{-1}\, \gamma_+(z)$ Birkhoff factorization

$\gamma_- : \mathbb{P}^1(\mathbb{C}) \setminus \{0\} \to G(\mathbb{C})$ $\quad \gamma_-(\infty) = 1$

holom.

$\gamma_+ : \Delta \longrightarrow G(\mathbb{C})$

$$\boxed{\begin{array}{c} \phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, K) \\ \Updownarrow \\ \gamma : \Delta^* \longrightarrow G(\mathbb{C}) \end{array}}$$

where $K = \mathbb{C}\{z\}[z^{-1}]$
$= \mathbb{C}(\{z\})$

field of convergent Laurent series

= germs of meromorphic functions at $z=0$

<u>Note</u>: $G(K) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, K)$

$\uparrow f : \Delta^* \to \mathbb{C}$ merom.

$G(\mathbb{C}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, \mathbb{C})$

$\gamma : \Delta^* \longrightarrow G(\mathbb{C})$

$\mathcal{O} = \mathbb{C}\{z\}$ convergent power series subring of $K = \mathbb{C}(\{z\})$

$\mathcal{Q} = z^{-1}\mathbb{C}[z^{-1}]$ divergent part (polar part)
Laurent polynomial regular at ∞

$\tilde{\mathcal{Q}} = \mathbb{C}[z^{-1}]$ unital

$\phi_+ \in G(\mathcal{O}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, \mathcal{O}) \iff \gamma_+ : \Delta \to G(\mathbb{C})$

$\phi_- \in G(\tilde{\mathcal{Q}}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, \tilde{\mathcal{Q}}) \iff \gamma_- : \mathbb{P}^1(\mathbb{C}) \setminus \{0\} \to G(\mathbb{C})$

$\varepsilon_- : \tilde{\mathcal{Q}} \to \mathbb{C}$ augmentation

counit in $\mathcal{H}$

$\varepsilon_- \circ \phi_- = \varepsilon \iff \gamma_-(\infty) = 1$

Birkhoff factorization property: given
$\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, K) = G(K)$
$\exists\ \phi_+ \in G(\mathcal{O}),\ \phi_- \in G(\tilde{\mathcal{Q}})$ w $\varepsilon_- \circ \phi_- = \varepsilon$
such that

$$\boxed{\phi = (\phi_- \circ S) * \phi_+}$$

Thm (Connes-Kreimer)

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n \qquad \mathcal{H}_0 = \mathbb{C}$$

For all $\phi \in G(K) = \mathrm{Hom}_A(\mathcal{H}, K) \quad \exists\, \phi_+, \phi_-$

$$\phi = (\phi_- \circ S) * \phi_+$$

ϕ_+, ϕ_- given by the recursive formula

polar part of Laurent series

$$(*) \begin{cases} \phi_-(x) = -T\left(\phi(x) + \sum \phi_-(x')\,\phi(x'')\right) \\ \phi_+(x) = \phi(x) + \phi_-(x) + \sum \phi_-(x')\,\phi(x'') \end{cases}$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$

lower deg terms

Proof: 1) first need to show that ϕ_- defined as in $(*)$ is an algebra homomorphism

$$\phi_- \in \mathrm{Hom}_A(\mathcal{H}, \mathbb{Q})$$

i.e. that $\phi_-(xy) = \phi_-(x)\,\phi_-(y)$

This depends crucially on the fact that projection onto polar part of a Laurent series is a Rota-Baxter operator i.e.

$$T(f)\,T(h) = -T(fh) + T(T(f)h) + T(f\,T(h))$$

(i.e. behaves somewhat like integration by parts)

Notice then that in the formula

$$\phi_+(x) = \phi(x) + \phi_-(x) + \Sigma\, \phi_-(x')\, \phi(x'')$$

the right hand side is

$$\langle \phi_- \otimes \phi, \Delta(X) \rangle = (\phi_- * \phi)(x)$$

So $\quad \phi_+ = \phi_- * \phi \quad$ hence

$$\phi = \phi_-^{-1} * \phi_+ = (\phi_- \circ S) * \phi_+$$

$\Rightarrow$ BPHZ $=$ Birkhoff factorization

Take $\boxed{\phi = U}$ i.e. on generators

$x = \Gamma$ of $\mathcal{H}(T) \qquad \phi(\Gamma) = U(\Gamma)$

unrenormalized Feynman integral (as Laurent series)

then formulae $\circledast$ for Birkhoff factorization <u>same</u> as BPHZ formulae!

$\phi_-(x) = -T(\phi(x) + \Sigma\, \phi_-(x')\, \phi(x''))$	$C(\Gamma) = -T(\underbrace{U(\Gamma) + \Sigma\, C(\gamma) U(\Gamma/\gamma)}_{\bar{R}(\Gamma)})$
$\phi_+(x) = \phi(x) + \phi_-(x) + \Sigma\, \phi_-(x')\, \phi(x'')$	$R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma)$

with $\boxed{\phi_- = C} \qquad \boxed{\phi_+ = R}$

Renormalization group

Grading $\bigoplus_{n\geq 0} \mathcal{H}_n$ on the Hopf algebra $\mathcal{H} = \mathcal{H}(T)$ of Feynman graphs gives 1-parameter family of automorphisms

$$\theta_t \in \mathrm{Aut}(G(T)) \qquad \forall t \in \mathbb{G}_a(\mathbb{C}) = \mathbb{C}$$

$$\frac{d}{dt}\theta_t\Big|_{t=0} = Y \qquad \text{grading operator} \qquad Y(x) = nX \quad x \in \mathcal{H}_n$$

$$\theta_t(X) = \exp(nt)\cdot X \qquad \forall X \in \mathcal{H}_n^\vee(T)$$

on dual Hopf alg hence on Lie alg of G

$$\langle \theta_t(u), x\rangle = \langle u, \theta_t(x)\rangle$$

defines action on dual Hopf algebra $u \in \mathcal{H}^\vee$ $x \in \mathcal{H}$

Action on loops $\gamma(z)$

Notice : energy scale dependence

$$\gamma(z) \longleftrightarrow \phi \in \mathrm{Hom}(\mathcal{H}, K)$$

$$\phi(\Gamma)(z) = U^z(\Gamma)$$

But $U^z_\mu(\Gamma, p_1, \ldots, p_N) = \int d^{D-z}k_1 \cdots d^{D-z}k_L \, \mu^{zL} \, I_\Gamma(p_1, \ldots, p_N, k_1, \ldots, k_L)$

energy scale

So $\gamma(z) = \gamma_\mu(z)$

<u>Proposition</u>: (CK)

(1) $$\boxed{\theta_{tz}(\gamma_\mu(z)) = \gamma_{e^t\mu}(z)}$$

Action of grading $\rightsquigarrow$ scaling of mass parameter

<u>Pf</u>: check on generators Γ:

$$\theta_{tz}(\gamma_\mu(z))(\Gamma) = \exp(b_1(\Gamma)\cdot tz)\, U_\mu^z(\Gamma)$$

$$= \exp(b_1(\Gamma)\cdot tz)\, \mu^{z\, b_1(\Gamma)} \int d^{D-z}k_1 \cdots d^{D-z}k_L\, I_\Gamma(p_1 \cdots p_N, k_1 \cdots k_L)$$

$$L = b_1(\Gamma)$$

$$= (e^t\mu)^{b_1(\Gamma) z} \int d^{D-z}k_1 \cdots d^{D-z}k_L\, I_\Gamma(p_i, k_i) = \gamma_{e^t\mu}(z)(\Gamma)$$

(2) $$\boxed{\frac{\partial}{\partial\mu}\gamma_{\mu -}(z) = 0}$$ in the Birkhoff factorization

$$\gamma_\mu(z) = \gamma_{\mu -}(z)^{-1}\, \gamma_{\mu +}(z)$$

<u>Pf</u>: Counterterms $C(\Gamma) = \phi_-(\Gamma)$

- depend polynomially on p^2 (external momenta)
  (recall discussion on BPHZ and local terms)
- Dimensional analysis $\rightsquigarrow$ depend polynomially on mass parameters m
  (for DimReg+MS not eg. for on-shell)
- Laurent series expansion of loop $\gamma_\mu(z)$: μ^{zL} dependence on μ $\rightsquigarrow$ $\log\mu$ powers
- Dimensional reasons $\log(\frac{\mu^2}{p^2})$ or $\log(\frac{\mu^2}{m^2})$ $\Rightarrow$ No μ dependence

Summarize: Data of perturbative renormalization in CK theory

- Given $\mathcal{T}$ (Lagrangian etc.)

  $\mathcal{H}(\mathcal{T})$ Hopf algebra of Feynman graphs

- Dual to affine group scheme $G_{\mathcal{T}}$

- Unrenormalized values of Feynman graphs $U_\mu^z(\Gamma)$ define loop $\gamma_\mu(z)$ in $G_{\mathcal{T}}(\mathbb{C})$

- BPHZ is Birkhoff factorization

$$\gamma_\mu(z) = \gamma_-^{-1}(z)\,\gamma_{\mu,+}(z)$$

γ_- = counterterms $\gamma_{\mu,+}(0)$ = renormalized values of all Feynman graphs

- Scaling property

$$\theta_{tz}(\gamma_\mu(z)) = \gamma_{e^t\mu}(z)$$

$$\frac{\partial}{\partial\mu}\gamma_-(z) = 0$$

- Renormalization group action

$$\mu\frac{\partial}{\partial\mu} \quad \text{on } \gamma_{\mu,+}(0)$$

Better description of Renormalization group

Thm :

- $\gamma_-(z)\ \theta_{tz}(\gamma_-(z)^{-1})$ regular at $z=0$

- $F_t := \lim_{z\to 0} \gamma_-(z)\, \theta_{tz}(\gamma_-(z)^{-1})$

  is 1-parameter subgroup of $G_J(\mathbb{C})$

  i.e. $F_t F_s = F_{t+s}$

- $F_t(x)$ polynomial in t for all $x \in \mathcal{H}$

  ($F_t \in \mathrm{Hom}_A(\mathcal{H}, \mathbb{C})$)

- $F_t\, \gamma_{\mu+}(0) = \gamma_{e^t\mu\,+}(0)$

- $\mathrm{Res}_{z=0}(\gamma_\mu(z)) := -\left(\frac{\partial}{\partial u}\gamma_-\left(\frac{1}{u}\right)\right)\Big|_{u=0}$

  $\beta := Y\ \mathrm{Res}(\gamma)$ Beta function

  $\beta \in \mathrm{Lie}(G_J)$

  then
  $$\frac{d}{dt}F_t\Big|_{t=0} = \beta$$

  infinitesimal generator of the renormalization group