Universal singular frame

$$\gamma_{U}(z,v) := T\, e^{-\frac{1}{z}\int_0^v u^Y(e)\,\frac{du}{u}}$$

where $\mathbf{e} = \sum_{n=1}^{\infty} e_{-n}$ generators of $Lie(U)$

(the universal source of counterterms for all physical theories)

Lemma: $$\gamma_{U}(-z,v) = \sum_{\substack{n\geq 0\\ k_j>0}} \frac{e_{-k_1} e_{-k_2} \cdots e_{-k_n}}{k_1\,(k_1+k_2)\cdots(k_1+k_2+\cdots+k_n)}\, v^{\sum k_i} z^{-n}$$

~~[illegible]~~

using $$T e^{\int_a^b \alpha(t)\,dt} = 1 + \sum_{n\geq 1} \int_{a\leq s_1\leq \cdots \leq s_n\leq b} \alpha(s_1)\cdots\alpha(s_n)\, ds_1\cdots ds_n$$

get $$v^{\sum k_i} z^{-n} \int_{0\leq s_1\leq\cdots\leq s_n\leq 1} s_1^{k_1-1}\cdots s_n^{k_n-1}\, ds_1\cdots ds_n$$ which gives coeff

as coeff of

$e_{-k_1}\cdots e_{-k_n}$

Lemma: $\Theta = [V,\nabla]$ an object of $\mathcal{E}$ (flat equising. vector bundle with W-equivalence class of connections)

$\Rightarrow \exists!$ representation

$\rho = \rho_\Theta$ of $U^* = U \rtimes \mathbb{G}_m$ on V

s.t. $\rho|_{\mathbb{G}_m}$ is the $\mathbb{Z}$-grading of V and

$D\rho(\gamma_U) \simeq \nabla$ with γ_U = univ. singular frame

Proof: Let G_V = group of linear transf. $S \subset \cancel{\mathrm{Aut}}(V)$ compatible w/ filtration $SW_{-n}(V) \subset W_{-n}(V)$ and inducing identity on graded $S|_{Gr_n^W} = id$

G_V is a unipotent algebraic group (non-canonically) isomorphic to upper-triangular matrices

Then ∇ defines a flat equisingular G_V-valued connection ω compatibly with equivalence relation

Then from previous results know that

$$\omega \sim D\left(Te^{-\frac{1}{2}\int_0^v u^Y(\beta)\frac{du}{u}}\right)$$

for some element $\beta \in \mathrm{Lie}(G_V)$

According to the grading of V, β decomposes as

$$\beta = \bigoplus_n \beta_n \quad \text{with } Y(\beta_n) = n\,\beta_n$$

This element β and its decomposition by grading determines a representation $\rho : \mathbb{U} \to G_V$ compatibly w/ gradings

i.e. $\rho : \mathbb{U} \rtimes \mathbb{G}_m \to G_V \rtimes \mathbb{G}_m$

by setting (at the level of the Lie algebra)

$$e_{-n} \longmapsto \beta_n$$

By construction this representation satisfies $D\rho(\gamma_{\nabla}) = D(Te^{-\frac{1}{2}\int_0^v u^Y(\beta)\frac{du}{u}}) \simeq \nabla$

Conversely: suppose given ρ repres of $\mathbb{U}^*$ on V compat. w/ grading

let $\gamma(z,v) = Te^{-\frac{1}{2}\int_0^v u^Y(\rho(e))\frac{du}{u}}$

$D\gamma$ is a flat equisingular connection

$\Rightarrow$ W-connection ∇ on $E^0 = B^0 \times V$ with right properties

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Lemma: $\Theta = [V, \nabla]$ in $Obj(\mathcal{E})$

(1) • For any $S \in Aut(V)$ compatible w/ grading $S\nabla S^{-1}$ equisingular connection

(2) • $\rho = \rho_{\Theta}$ as above satisfies

$$\rho_{[V, S\nabla S^{-1}]} = S\rho_{[V,\nabla]}S^{-1}$$

(3) • $\nabla \sim S\nabla S^{-1}$ iff $[\rho_{[V,\nabla]}, S] = 0$ (commutator)

Pf: use

$$S\,Te^{-\frac{1}{2}\int_0^v u^Y(\beta)\frac{du}{u}}S^{-1} = Te^{-\frac{1}{2}\int_0^v u^Y(S\beta S^{-1})\frac{du}{u}}$$

(3): same $\beta \Leftrightarrow \beta S = S\beta$

Compatibility of morphisms:

Prop:
$\Theta = [V, \nabla] \quad \Theta' = [V', \nabla'] \in Obj(\mathcal{E})$

$T : V \to V'$ compatible w/ grading

then
$T \in Hom_{\mathcal{E}}(\Theta, \Theta')$ iff

$$T\rho_{\Theta} = \rho_{\Theta'} T$$

Pf: $S \in Aut(E' \oplus E)$ unipotent automorphism

- compatible w/ grading $\quad S = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad S \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix} S^{-1} \sim \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix}$ iff

$$\begin{pmatrix} \beta' & 0 \\ 0 & \beta \end{pmatrix} S = S \begin{pmatrix} \beta' & 0 \\ 0 & \beta \end{pmatrix}$$

$$\Longleftrightarrow \quad \beta' T = T \beta$$

$$\mathbb{T} e^{-\frac{1}{z}\int_0^v u^Y(\beta') \frac{du}{u}} \; T = T \; \mathbb{T} e^{-\frac{1}{z}\int_0^v u^Y(\beta) \frac{du}{u}}$$

hence $\rho_{\Theta'} T = T \rho_{\Theta}$

Also check compatibility w/ tensor products (5)

$$(V, \nabla) \otimes (V', \nabla') = (V \otimes V', \nabla \otimes 1 + 1 \otimes \nabla')$$

(compatibly w/ the W-equivalence relation)

$$T e^{-\frac{1}{2}\int_0^v u^Y(\beta\otimes 1 + 1\otimes\beta')\frac{du}{u}}$$

$$= T e^{-\frac{1}{2}\int_0^v u^Y(\beta)\frac{du}{u}} \otimes T e^{-\frac{1}{2}\int_0^v u^Y(\beta')\frac{du}{u}}$$

(for morphisms check on $T\otimes 1$ and $1 \otimes T$) $\qquad \rho_{\Theta\otimes\Theta'} = \rho_\Theta \otimes \rho_{\Theta'}$

Completes proof that

$$\mathrm{Rep}_{U^*} \simeq \mathcal{E} \qquad \text{(equivalence of categories)}$$

General fact: (neutral) Tannakian categories $\Leftrightarrow$ Rep_G for some affine group scheme G

(neutral) Tannakian category: $\mathcal{C}$

- abelian category
- k-linear tensor category, rigid
- additive exact tensor functor $\omega: \mathcal{C} \to \mathrm{Vect}_k$
- neutral if same k (in general $\mathrm{Vect}_{k'}$, k' extension of k)

$\Rightarrow \mathcal{C} \simeq \mathrm{Rep}_G \qquad G = \mathrm{Aut}^{\otimes}(\omega)$ automorphisms of