

Quantum statistical mechanics
of \mathbb{Q} -lattices

Based on:

- Alain Connes and M.M., *Quantum statistical mechanics of \mathbb{Q} -lattices*, math.NT/0404128
- Alain Connes, Niranjana Ramachandran, and M.M., *KMS states and complex multiplication*, preprint 2004

Unified setting for:

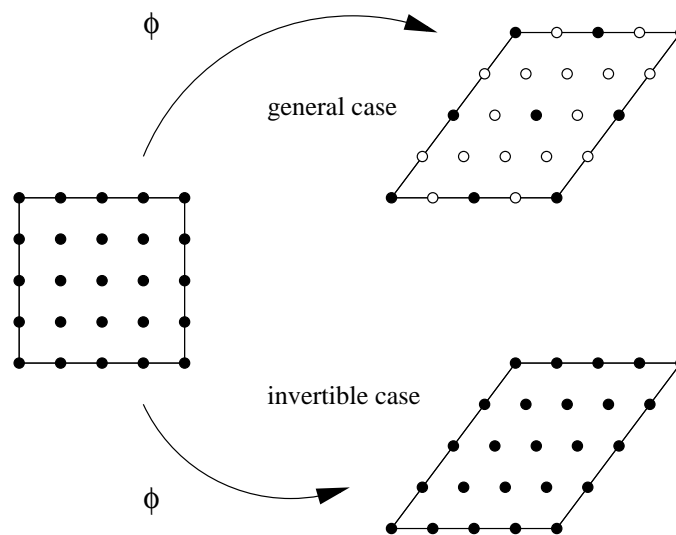
- Modular Hecke algebra (Connes–Moscovici)
- Spectral realization of zeros of $\zeta(s)$ (Connes)
- Bost–Connes system with arithmetic spontaneous symmetry breaking
- NC boundary of modular curves (Manin–M)

Q-lattices (Λ, ϕ) Q-lattice in \mathbb{R}^n

lattice $\Lambda \subset \mathbb{R}^n$ + labels of torsion points

$$\phi : \mathbb{Q}^n / \mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda / \Lambda$$

group homomorphism (invertible Q-lat is isom)



Commensurability

$$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$$

iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and

$$\phi_1 = \phi_2 \pmod{\Lambda_1 + \Lambda_2}$$

Q-lattices / Commensurability \Rightarrow NC space

Quantum Statistical Mechanics

\mathcal{A} = algebra of observables (C^* -algebra)

State: $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ linear

$$\varphi(1) = 1, \quad \varphi(a^*a) \geq 0$$

Time evolution $\sigma_t \in \text{Aut}(\mathcal{A})$

(rep on Hilbert space \mathcal{H})

$$\text{Hamiltonian} \quad H = \frac{d}{dt} \sigma_t|_{t=0}$$

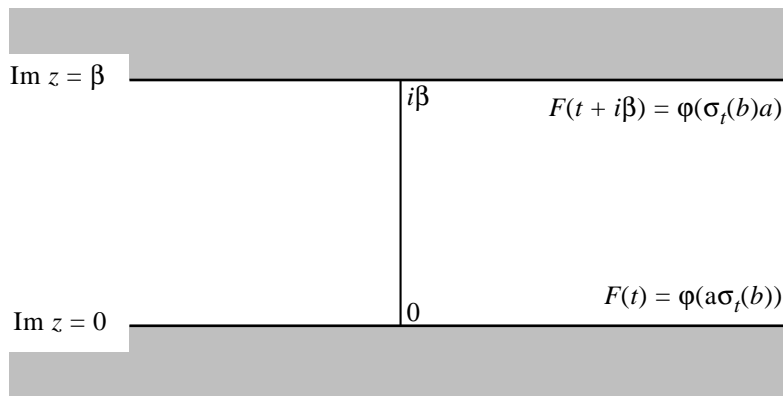
Equilibrium state (inverse temperature $\beta = 1/kT$)

$$\frac{1}{Z(\beta)} \text{Tr} (a e^{-\beta H}) \quad Z(\beta) = \text{Tr} (e^{-\beta H})$$

KMS states $\varphi \in \text{KMS}_\beta$ ($0 < \beta < \infty$)

$\forall a, b \in \mathcal{A} \exists$ holom function $F_{a,b}(z)$ on strip: $\forall t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$$



Ground states ($\beta = \infty, T = 0$)

At $T > 0$ simplex $\text{KMS}_\beta \rightsquigarrow$ extremal \mathcal{E}_β
(Points on NC space \mathcal{A})

At $T = 0$: $\text{KMS}_\infty =$ weak limits of KMS_β

$$\varphi_\infty(a) = \lim_{\beta \rightarrow \infty} \varphi_\beta(a)$$

Symmetries \rightsquigarrow action on \mathcal{E}_β

Automorphisms $G \subset \text{Aut}(\mathcal{A}), g\sigma_t = \sigma_t g$

Mod Inner: $u = \text{unitary}$ $\sigma_t(u) = u$

$$a \mapsto uau^*$$

Endomorphisms $\rho\sigma_t = \sigma_t\rho$ $e = \rho(1)$

For $\varphi(e) \neq 0$

$$\rho^*(\varphi) = \frac{1}{\varphi(e)} \varphi \circ \rho$$

Mod Inner: $u = \text{isometry}$ $\sigma_t(u) = \lambda^{it}u$

Action: (on \mathcal{E}_∞ : warming up/cooling down)

$$W_\beta(\varphi)(a) = \frac{\text{Tr}(\pi_\varphi(a) e^{-\beta H})}{\text{Tr}(e^{-\beta H})}$$

The Bost–Connes system

J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, *Selecta Math.* (1995)

$r \in \mathbb{Q}/\mathbb{Z} \rightsquigarrow$ “phase operators” $e(r)$

Fock space $e(r)|n\rangle = \alpha(\zeta_r^n)|n\rangle$

$\zeta_{a/b} = \zeta_b^a$ abstract roots of 1 embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$

Optical phase: $|\theta_{m,N}\rangle = e\left(\frac{m}{N+1}\right) \cdot v_N$ superposition of occupation states $v_N = \frac{1}{(N+1)^{1/2}} \sum_{n=0}^N |n\rangle$

$n \in \mathbb{N}^\times = \mathbb{Z}_{>0} \rightsquigarrow$ changes of scale μ_n

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

1-dimensional \mathbb{Q} -lattices

$$(\Lambda, \phi) = (\lambda \mathbb{Z}, \lambda \rho) \quad \lambda > 0$$

$$\rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$$

Up to scaling λ : algebra $C(\hat{\mathbb{Z}}) \simeq C^*(\mathbb{Q}/\mathbb{Z})$

Commensurability Action of \mathbb{N}^\times

$$\alpha_n(f)(\rho) = f(n^{-1}\rho)$$

1-dimensional \mathbb{Q} -lattices / Commensurability

$$\Rightarrow \text{NC space } C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times$$

Hecke algebra $\mathcal{A} = \mathcal{A}_1$

$(\Gamma_0, \Gamma) = (P_{\mathbb{Z}}, P_{\mathbb{Q}})$ $ax + b$ group

$f : \Gamma_0 \backslash \Gamma / \Gamma_0 \rightarrow \mathbb{C}$ (or to \mathbb{Q})

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f_1(\gamma \gamma_1^{-1}) f_2(\gamma_1)$$

$$f^*(\gamma) := \overline{f(\gamma^{-1})}$$

Regular rep $\ell^2(\Gamma_0 \backslash \Gamma) \Rightarrow$ von Neumann alg
 \leadsto time evolution

$\gamma \in \Gamma$: $L(\gamma) = \#\Gamma_0 \gamma$ and $R(\gamma) = L(\gamma^{-1})$

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)} \right)^{-it} f(\gamma)$$

$$\sigma_t(\mu_n) = n^{it} \mu_n \quad \sigma_t(e(r)) = e(r)$$

Representations Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^\times)$

$\alpha \in \hat{\mathbb{Z}}^* = \text{GL}_1(\hat{\mathbb{Z}}) \Leftrightarrow$ embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$

$$\pi_\alpha(e(r)) \epsilon_k = \alpha(\zeta_r^k) \epsilon_k$$

$$\pi_\alpha(\mu_n) \epsilon_k = \epsilon_{nk}$$

Hamiltonian $H \epsilon_k = \log k \epsilon_k$

$$Z(\beta) = \text{Tr} \left(e^{-\beta H} \right) = \sum_{k=1}^{\infty} k^{-\beta} = \zeta(\beta)$$

Partition function = Riemann zeta function

Structure of KMS states (Bost–Connes)

- $\beta \leq 1 \Rightarrow$ unique KMS_β state
- $\beta > 1 \Rightarrow \mathcal{E}_\beta \cong \widehat{\mathbb{Z}}^*$ (free transitive action)

$$\varphi_{\beta,\alpha}(x) = \frac{1}{\zeta(\beta)} \text{Tr} \left(\pi_\alpha(x) e^{-\beta H} \right)$$

- $\beta = \infty$ Galois action $\theta : \text{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \xrightarrow{\cong} \widehat{\mathbb{Z}}^*$

$$\gamma \varphi(x) = \varphi(\theta(\gamma) x)$$

$$\forall \varphi \in \mathcal{E}_\infty \quad \forall \gamma \in \text{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$$

and $\forall x \in \mathcal{A}_\mathbb{Q}$ (arithmetic subalgebra)

$$\varphi(\mathcal{A}_\mathbb{Q}) \subset \mathbb{Q}^{cycl}$$

NCG and class field theory

\mathbb{K} = number field $[\mathbb{K} : \mathbb{Q}] = n$, an algebraic closure $\bar{\mathbb{K}} \rightsquigarrow$ group of symmetries $\text{Gal}(\bar{\mathbb{K}}/\mathbb{K})$

Max abelian extension \mathbb{K}^{ab} :

$$\text{Gal}(\mathbb{K}^{ab}/\mathbb{K}) = \text{Gal}(\bar{\mathbb{K}}/\mathbb{K})^{ab}$$

Kronecker–Weber: $\mathbb{K} = \mathbb{Q}$

$$\mathbb{Q}^{ab} = \mathbb{Q}^{cycl} \quad \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*$$

Hilbert 12th problem (explicit class field theory)

Generators of \mathbb{K}^{ab} + action of $\text{Gal}(\mathbb{K}^{ab}/\mathbb{K})$

Solved for: \mathbb{Q} and $\mathbb{Q}(\sqrt{-d})$ (imaginary quadratic)

Question: Can NCG say something new?

(at least for real quadratic $\mathbb{Q}(\sqrt{d})$)

Bost–Connes system \rightsquigarrow Kronecker–Weber

adèlic description: $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ and $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$

$$\mathcal{E}_\infty \simeq \mathrm{GL}_1(\mathbb{Q}) \backslash \mathrm{GL}_1(\mathbb{A}) / \mathbb{R}_+^*$$

Number field \mathbb{K} with $[\mathbb{K} : \mathbb{Q}] = n$

$$\mathbb{K}^* \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$$

\Rightarrow Seek extension of Bost–Connes system to $\mathrm{GL}_n(\mathbb{A})$

GL_2 case \rightsquigarrow Modular curves, modular forms

(A. Connes, M.M. *Quantum Statistical Mechanics of \mathbb{Q} -lattices*)

2-dimensional \mathbb{Q} -lattices

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}\tau), \lambda\rho)$$

$$\lambda \in \mathbb{C}^*, \tau \in \mathbb{H}, \rho \in M_2(\hat{\mathbb{Z}}) = \text{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$$

Up to scale $\lambda \in \mathbb{C}^*$ and isomorphism

$$M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \quad \text{mod } \Gamma = \text{SL}(2, \mathbb{Z})$$

Commensurability action of $\text{GL}_2^+(\mathbb{Q})$ (partially defined)

\Rightarrow NC space

Functions on

$$\mathcal{U} = \{(g, \rho, z) \in \text{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \mid g\rho \in M_2(\hat{\mathbb{Z}})\}$$

invariant under $\Gamma \times \Gamma$ action $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 z)$

Hecke algebra $\mathcal{A} = \mathcal{A}_2$

convolution product

$$(f_1 * f_2)(g, \rho, z) = \sum_{s \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q}) : s\rho \in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\rho, s(z)) f_2(s, \rho, z)$$

$$f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}$$

Time evolution:

$$\sigma_t(f)(g, \rho, z) = \det(g)^{it} f(g, \rho, z)$$

Hilbert spaces: for $\rho \in M_2(\hat{\mathbb{Z}})$

$$G_\rho = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}})\}$$

$$\mathcal{H}_\rho = \ell^2(\Gamma \backslash G_\rho)$$

Representations:

$L = (\Lambda, \phi) = (\rho, z)$ gives representation on \mathcal{H}_ρ :

$$(\pi_L(f)\xi)(g) = \sum_{s \in \Gamma \backslash G_\rho} f(gs^{-1}, s\rho, s(z)) \xi(s)$$

$L = (\Lambda, \phi)$ invertible $\Rightarrow \mathcal{H}_\rho \cong \ell^2(\Gamma \backslash M_2^+(\mathbb{Z}))$

Hamiltonian

$$H \epsilon_m = \log \det(m) \epsilon_m$$

\Rightarrow positive energy

Partition function $\sigma(k) = \sum_{d|k} d$

$$\begin{aligned} Z(\beta) &= \text{Tr} \left(e^{-\beta H} \right) = \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \det(m)^{-\beta} \\ &= \sum_{k=1}^{\infty} \sigma(k) k^{-\beta} = \zeta(\beta) \zeta(\beta - 1) \end{aligned}$$

Structure of KMS states GL_2 -system

- $\beta \leq 1$ No KMS states
- $1 < \beta < 2$ Unique ?
- $\beta > 2 \Rightarrow \mathcal{E}_\beta \cong GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \mathbb{C}^*$

$$\varphi_{\beta,L}(f) = \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} f(1, m\rho, m(z)) \det(m)^{-\beta}$$

$$L = (\rho, z)$$

(Hard to show: all $\varphi \in \mathcal{E}_\beta$ of this form)

Two phase transitions

Symmetries $2 < \beta \leq \infty$

$GL_2(\mathbb{A}_f) = GL_2^+(\mathbb{Q})GL(\widehat{\mathbb{Z}})$ acts on \mathcal{A}_2

- $GL(\widehat{\mathbb{Z}})$ by automorphisms (deck transformations of coverings of modular curves)

$$\theta_\gamma(f)(g, \rho, z) = f(g, \rho\gamma, z)$$

- $GL_2^+(\mathbb{Q})$ by endomorphisms

$$\theta_m(f)(g, \rho, z) = \begin{cases} f(g, \rho\tilde{m}^{-1}, z) & \rho \in m M_2(\widehat{\mathbb{Z}}) \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{m} = \det(m)m^{-1}$$

$\mathbb{Q}^* \hookrightarrow GL_2(\mathbb{A}_f)$ acts by inner

$\Rightarrow S = \mathbb{Q}^* \backslash GL_2(\mathbb{A}_f)$ symmetries on \mathcal{E}_β

Action of $GL_2^+(\mathbb{Q})$ on \mathcal{E}_∞ : warming/cooling

Arithmetic subalgebra

Look back at $\mathcal{A}_{\mathbb{Q}} = \mathcal{A}_{1,\mathbb{Q}}$ (Bost–Connes case):
homogeneous functions of weight zero on 1-dim \mathbb{Q} -lattices

$$\epsilon_{k,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-k}$$

normalized by covolume $e_{k,a} := c^k \epsilon_{k,a}$
($c(\Lambda) \sim |\Lambda|$ covolume with $(2\pi\sqrt{-1})c(\mathbb{Z}) = 1$)

GL_2 case $\mathcal{A}_{\mathbb{Q}} = \mathcal{A}_{2,\mathbb{Q}}$: Eisenstein series

$$E_{2k,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-2k}$$

and

$$X_a(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-2} - \sum'_{y \in \Lambda} y^{-2}$$

normalized to weight zero

\Rightarrow Modular functions

The modular field

Weierstrass \wp -function:

parameterization $w \mapsto (1, \wp(w; \tau, 1), \frac{d}{dw}\wp(w; \tau, 1))$

of elliptic curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ by $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$

Fricke functions: (homog weight zero) $v \in \mathbb{Q}^2/\mathbb{Z}^2$

$$f_v(z) = -2^7 3^5 \frac{g_2(z)g_3(z)}{\Delta(z)} \wp(\lambda_z(v); z, 1)$$

$\Delta(z) = g_2^3 - 27g_3^2$ discriminant, $\lambda_z(v) := v_1z + v_2$

Generators of the modular field F

(Shimura) $\text{Aut}(F) \cong \mathbb{Q}^* \backslash \text{GL}_2(\mathbb{A}_f)$

$\tau \in \mathbb{H}$ generic \Rightarrow evaluation $f \mapsto f(\tau)$

\rightsquigarrow embedding $F \hookrightarrow \mathbb{C}$ image F_τ

$$\theta_\tau : \text{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^* \backslash \text{GL}_2(\mathbb{A}_f)$$

Galois action on \mathcal{E}_∞

State $\varphi = \varphi_{\infty,L} \in \mathcal{E}_\infty$ ($L = (\rho, \tau)$ generic) \Rightarrow

$$\theta_\varphi : \text{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^* \backslash \text{GL}_2(\mathbb{A}_f)$$

$$\theta_\varphi(\gamma) = \rho^{-1} \theta_\tau(\gamma) \rho$$

$\varphi = \varphi_{\infty,L}$ with $L = (\rho, \tau)$

$\forall f \in \mathcal{A}_{2,\mathbb{Q}}$ and $\forall \gamma \in \text{Gal}(F_\tau/\mathbb{Q})$

$$\gamma \varphi(f) = \varphi(\theta_\varphi(\gamma)f)$$

$\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau$

Shimura varieties: $Sh(G, X) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X$

$$\begin{aligned} Sh(\mathrm{GL}_2, \mathbb{H}^\pm) &= \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \\ &= \mathrm{GL}_2^+(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H} = \mathrm{GL}_2^+(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}) / \mathbb{C}^* \end{aligned}$$

Components: $\pi_0(Sh(\mathrm{GL}_2, \mathbb{H}^\pm)) = Sh(\mathrm{GL}_1, \{\pm 1\})$

$$Sh(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q}) \backslash \mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\} = \mathbb{Q}_+^* \backslash \mathbb{A}_f^*$$

Commutative pts of NC space $\mathrm{GL}_1(\mathbb{Q}) \backslash \mathbb{A}_f$ Bost–Connes

$$\mathcal{A} \simeq C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^* \quad \text{Morita equiv}$$

Modular curves: $Sh(\mathrm{GL}_2, \mathbb{H}^\pm) =$ adèlic version of modular tower

$\varprojlim \Gamma \backslash \mathbb{H}$ congruence subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{Z}) \rightsquigarrow$
 congr subgr in $\mathrm{SL}(2, \mathbb{Q}) \rightsquigarrow$ components $Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$

NC Shimura varieties

Bost–Connes system:

$$\begin{aligned} Sh^{(nc)}(\{\pm 1\}, GL_1) &:= GL_1(\mathbb{Q}) \backslash (\mathbb{A}_f \times \{\pm 1\}) \\ &= GL_1(\mathbb{Q}) \backslash \mathbb{A}^\cdot / \mathbb{R}_+^* \end{aligned}$$

$$\mathbb{A}^\cdot := \mathbb{A}_f \times \mathbb{R}^*$$

Compactification:

$$\overline{Sh^{(nc)}}(\{\pm 1\}, GL_1) = GL_1(\mathbb{Q}) \backslash \mathbb{A} / \mathbb{R}_+^*$$

Dual space (crossed product by time evolution σ_t)

$$\mathcal{L} = GL_1(\mathbb{Q}) \backslash \mathbb{A} \rightarrow GL_1(\mathbb{Q}) \backslash \mathbb{A} / \mathbb{R}_+^*$$

\mathbb{R}_+^* -bundle

Spectral realization of zeros of ζ (Connes)

1-dim \mathbb{Q} -lattices (not up to scaling)

mod commensurability

GL₂-system:

$$Sh^{(nc)}(\mathbb{H}^\pm, GL_2) := GL_2(\mathbb{Q}) \backslash (M_2(\mathbb{A}_f) \times \mathbb{H}^\pm)$$

Compactification:

$$\begin{aligned} \overline{Sh}^{(nc)}(\mathbb{H}^\pm, GL_2) &:= GL_2(\mathbb{Q}) \backslash (M_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{C})) \\ &= GL_2(\mathbb{Q}) \backslash M_2(\mathbb{A}) / \mathbb{C}^* \end{aligned}$$

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{H}^\pm \cup \mathbb{P}^1(\mathbb{R})$$

⇒ Adding NC boundary of modular curves

Dual system: \mathbb{C}^* -bundle $GL_2(\mathbb{Q}) \backslash M_2(\mathbb{A})$

Modular forms (instead of modular functions)

⇒ Modular Hecke algebra (Connes–Moscovici)

Compatibility:

$$\det \times \text{sign} : Sh(\mathbb{H}^\pm, GL_2) \rightarrow Sh(\{\pm 1\}, GL_1)$$

passing to connected components π_0

Class field theory $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$

$\tau \in \mathbb{H}$ CM point $\mathbb{K} = \mathbb{Q}(\tau)$

Evaluation $F \rightarrow F_\tau \subset \mathbb{C}$ not embedding

$$F_\tau \simeq \mathbb{K}^{ab} \quad \{f(\tau), f \in F\} \text{ generators}$$

Galois action

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{K}^* & \longrightarrow & \mathrm{GL}_1(\mathbb{A}_{\mathbb{K},f}) & \xrightarrow{\simeq} & \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) \longrightarrow 1 \\ & & & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Q}^* & \longrightarrow & \mathrm{GL}_2(\mathbb{A}_f) & \xrightarrow{\simeq} & \mathrm{Aut}(F) \longrightarrow 1. \end{array}$$

$$\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_f \otimes \mathbb{K}$$

\Rightarrow Specialization of GL_2 -system at CM points
 \leadsto CFT for $\mathbb{K} = \mathbb{Q}(\tau)$

What about $\mathbb{Q}(\sqrt{d})$??

NC boundary of mod curves in $\overline{Sh}^{(nc)}(\mathbb{H}^\pm, \mathrm{GL}_2)$

(Manin's real multiplication program)