Non-commutative Geometry
Research Seminar 1/24/08
B. Field

Comments:

There are different metric conventions, physicists usually pick one and stick with it like a religion. We need a dictionary to get the signs correct.

see Appendix B.
Field Theory in Particle Physics
J. Smith & B. de Wit
0-444-86-996-4

take $c = \hbar = 1$ "God given units" actually Heaviside-Lorentz units

3 - conventions $\eta^{\mu\nu} = \pm\,\mathrm{diag}(-,+,+,+)$ "mostly positive" "mostly negative"
+ Pauli metric w/ imaginary 4th component (old-fashioned, but useful this is Euclidean space)

$x_\mu = (x, y, z, t)$
$p_\mu = (p_x, p_y, p_z, E)$

** I use $\eta^{\mu\nu}$ instead of $g^{\mu\nu}$ (flat!)

Clifford algebra $[\eta(-,+,+,+)]$

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ $\mu,\nu = 0,1,2,3$

$\gamma^{1,2,3} = \gamma_{1,2,3}$ $\gamma^0 = -i\gamma_4$

$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_1\gamma_2\gamma_3\gamma_4$

$\bar\psi \equiv i\psi^{*T}\gamma_0 = \psi^{*T}\gamma_4$

$[\eta(+,-,-,-)]$

$\gamma^{1,2,3} = i\gamma_{1,2,3}$ $\gamma^0 = -\gamma_4$

$\bar\psi = \psi^{*T}\gamma^0 = \psi^{*T}\gamma_4$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma_1\gamma_2\gamma_3\gamma_4$

Pauli	$\pm(-,+,+,+)$
$p_\mu x_\mu = \vec p\cdot\vec x - Et$	$p_\mu x^\mu = \pm(\vec p\cdot\vec x - Et)$
$\Box = \partial_\mu^2 = \dfrac{\partial^2}{\partial x_i^2} - \dfrac{\partial^2}{\partial t^2}$	$\Box = \partial_\mu\partial^\mu = \pm\left(\dfrac{\partial^2}{\partial x_i^2} - \dfrac{\partial^2}{\partial t^2}\right)$
$(\partial_\mu\phi)^2 = \left(\dfrac{\partial\phi}{\partial x_i}\right)^2 - \left(\dfrac{\partial\phi}{\partial t}\right)^2$	$\partial_\mu\phi\,\partial^\mu\phi = \pm\left(\left(\dfrac{\partial\phi}{\partial x_i}\right)^2 - \left(\dfrac{\partial\phi}{\partial t}\right)^2\right)$
$F_{ij} = \varepsilon_{ijk}B_k$	$F_{ij} = F^{ij} = \pm\varepsilon_{ijk}B_k$
$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$	$\mathcal{L}_0 = \mp\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$
$(\Box - m^2)\phi = 0$	$(\pm\Box - m^2)\phi = 0$
$\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}^2 + J_\mu A_\mu$	$\mathcal{L}_{Max} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \pm J_\mu A^\mu$
$\partial_\nu F_{\mu\nu} = J_\mu$	$\partial_\nu F^{\mu\nu} = \pm J^\mu$
fermions $\mathcal{L}_{1/2} = -\bar\psi\not{\partial}\psi - m\bar\psi\psi$	$\mathcal{L}_{1/2} = i\bar\psi\not{\partial}\psi - m\bar\psi\psi$
$(\not{\partial} + m)\psi = 0$	$(-i\not{\partial} + m)\psi = 0$
$(i\not{p} + m)u(p) = 0$	$(\not{p} - m)u(p) = 0$
$(i\not{p} - m)v(p) = 0$	$(\not{p} + m)v(p) = 0$

<u>Original</u> Phys Rev. <u>125</u> 1067 (1962)

A little about color $\equiv SU(3)$

$$[T^a, T^b] = i f_{abc} T^c$$

$$\{T^a, T^b\} = \tfrac{1}{3}\delta^{ab} + d_{abc} T^c$$

$$T^a = \frac{\lambda^a}{2}$$

abc	f_{abc}
123	1
147	1/2
156	−1/2
246	1/2
257	1/2
345	1/2
367	−1/2
458	$\sqrt{3}/2$
678	$\sqrt{3}/2$

abc		d_{abc}	
118	366	$\sqrt{3}/3$	−
146	377	1/2	−
157	448	1/2	$-\sqrt{3}/$
228	558	$\sqrt{3}/3$	$-\sqrt{3}/$
247	668	−1/2	$-\sqrt{3}/$
256	778	1/2	$-\sqrt{3}/$
338	888	$\sqrt{3}/3$	$-\sqrt{3}/$
344		1/2	
355		1/2	

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$f_{abe}f_{ecd} + f_{cbe}f_{aed} + f_{dbe}f_{ace} = 0$$ Jacobi identity (cyclic trace condition)

fundamental t^a

adjoint T^a $\qquad Tr(t^a t^b) = T_R\,\delta^{ab} \qquad T_R = \frac{1}{2}$

$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac} \qquad C_F = \frac{N^2-1}{2N} \qquad SU(N)$$

$$Tr(T^C T^D) = \sum_{A,B} f^{ABC} f^{ABD} = C_A\,\delta^{CD}, \quad C_A = N$$

$SU(3)$ $C_F = 4/3$, $C_A = 3$

Now then, scalar field theory [Euclidean] This theory has problem w/ gauge invariance

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 + J\phi$$

eqn of motion: $(\Box - m^2)\phi(x) = -J(x)$ "Klein-Gordon theory"

Green's fcn $(\Box - m^2)\Delta(x) = i\delta^4(x)$ $\delta^4(x) = (2\pi)^{-4}\int d^4k\, e^{ik\cdot x}$

$$\Rightarrow \Delta(x) = \frac{1}{i(2\pi)^4}\int d^4k\, \frac{e^{ik\cdot x}}{k^2+m^2}$$

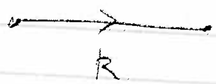
for an on-shell particle, this is not well defined. Poles @ $k_0 = \pm\omega(\vec{k})$

$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$

//

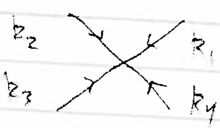
$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 - \lambda\phi^3 - g\phi^4$$

$\Delta(x-y)$ connects space-time points x and y

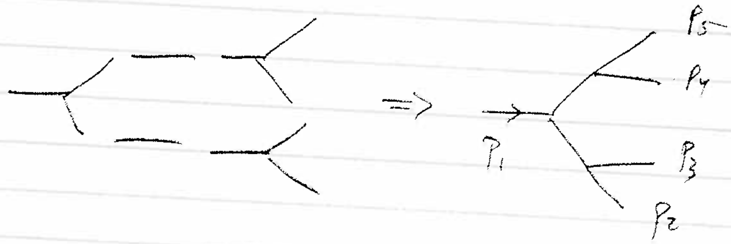


$\dfrac{1}{i(2\pi)^4}\;\dfrac{1}{k^2+m^2-i\epsilon}$

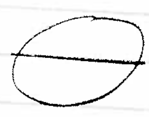
$i(2\pi)^4\,\delta^4(k_1+k_2+k_3)\,(-\lambda)$



$i(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\,(-g)$

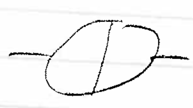


-6λ



$3\lambda^2$

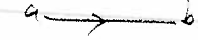
$18\lambda^2$



$648\lambda^4$

Let's try pion scattering

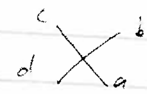
$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}\mu^2\sigma^2$$
$$-\lambda\sigma\phi^2 - g(\phi^2)^2 - g'\sigma^2\phi^2 - \lambda'\sigma^3 - g''\sigma^4$$



$\frac{1}{i(2\pi)^4}\ \frac{\delta^{ab}}{k^2+m^2}$


$\frac{1}{i(2\pi)^4}\ \frac{1}{k^2+\mu^2}$

$i(2\pi)^4\delta^4()(-\lambda)\delta^{ab}$



$i(2\pi)^4\delta^4()(-g)(\delta^{ab}\delta^{cd}$

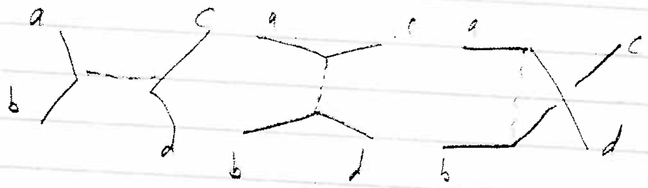
$i(2\pi)^4\delta^4()(-g')\delta^{ab}$


$i(2\pi)^4\delta^4(\cdot)(-\lambda')$


$i(2\pi)^4\delta^4(\dots)(-g'')$

$\pi\pi \to \pi\pi$


$-i(2\pi)^4\delta^4(\dots)\,8g\,(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})$



s, t, u

$$= i(2\pi)^4\delta^4(\dots)\,4\lambda^2\left\{\frac{\delta^{ab}\delta^{cd}}{(p^a+p^b)^2+\mu^2} + \frac{\delta^{ac}\delta^{bd}}{(p^a-p^c)^2+\mu^2} + \frac{\delta^{ad}\delta^{bc}}{(p^a-p^d)^2+\mu^2}\right\}$$

Mandelstam

$$s = -(p^a+p^b)^2 \quad t = -(p^a-p^c)^2 \quad u = -(p^a-p^d)^2$$

$$s+t+u = -(p^a)^2-(p^b)^2-(p^c)^2-(p^d)^2$$
$$= 4m^2$$

What we really want is $\mathcal{M}$

$$G = \left\{\prod_j\left(\int d^4k_j\, e^{ik_j\cdot x_j}\,\Delta^{(j)}(k_j)\right)\right\} i(2\pi)^4\delta^4(\dots)\,\mathcal{M}(k_i)$$

$\mathcal{M}$ is like the $\sqrt{\text{probability}}$

$1+2 \to 3+4$

$$\lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$$

"Källen" or triangular function

$$\frac{d\sigma}{dt} = \frac{1}{16\pi}\,\frac{|\mathcal{M}|^2}{\lambda(s,m_1^2,m_2^2)}$$

$$\Gamma(A\to B+C) = \frac{N}{16\pi}\,\frac{\lambda^{1/2}(M_A^2,M_B^2,M_C^2)}{M_A^3}\,|\mathcal{M}|^2$$

if $B=C$ $N=\frac{1}{2}$, else $N=1$

Let's talk spin. start with spin = 1 (Weak gauge bosons, gluons)

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)^2 - \frac{1}{2} m^2 V_\mu^{\,2}$$

$$= -\frac{1}{2}\left(\partial_\mu V_\nu\right)^2 + \frac{1}{2}\partial_\mu V_\nu \partial_\nu V_\mu - \frac{1}{2} m^2 V_\mu^{\,2}$$ "Proca Lagrangian"

$$(\Box - m^2) V_\mu - \partial_\mu \partial_\nu V_\nu = 0$$

$$(\Box - m^2) V_\mu = 0 \qquad \partial_\mu V_\mu = 0$$ is equivalent → gives three polarization

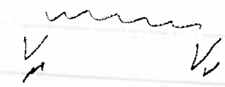
$$V_\mu(x) = \varepsilon_\mu(\vec{k}) e^{ik\cdot x} \qquad k^2 = -m^2$$

$$\Downarrow$$

$$k\cdot\varepsilon(\vec{k}) = 0$$ [this is a Ward-Takahashi identity]

$$\varepsilon(\vec{k},\lambda)\cdot\varepsilon(\vec{k},\lambda') = \delta_{\lambda\lambda'} \qquad \lambda,\lambda' = 1,2,3$$

$$\Delta_{\mu\nu}(k) = \frac{1}{i(2\pi)^4}\frac{1}{k^2+m^2}\left(\delta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\right)$$



$$M^{\lambda_1\cdots\lambda_n} = M_{\mu_1\cdots\mu_n}\varepsilon_{\mu_1}(\vec{k},\lambda_1)\cdots\varepsilon_{\mu_n}(\vec{k},\lambda_n)$$

$$M^\lambda = M_\mu \varepsilon_\mu(\vec{k},\lambda) \qquad \sum_{\lambda=1}^{3}\varepsilon_\mu\bar{\varepsilon}_\nu = \delta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}$$

There is a subtlty w/ massless spin-1 (gluons) (two-polarizations)

We have to add a gauge-fixing parameter to ment the the Green's function

$$\mathcal{L}_{g.f.} = -\frac{1}{2}\left(\lambda\,\partial_\mu A_\mu\right)^2$$

$$\Delta_{\mu\nu} = \frac{1}{i(2\pi)^4}\frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\lambda^{-2})\frac{k_\mu k_\nu}{k^2}\right)$$ λ can be renormalized!

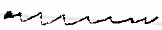
$$\sum_\lambda \varepsilon_\mu\bar{\varepsilon}_\nu = \delta_{\mu\nu} - \frac{k_\mu k_\nu^* + k_\mu^* k_\nu}{k\cdot k^*} \qquad k^2 = 0$$

$$\sum_\lambda |M_\mu \varepsilon_\mu|^2 = M_\mu \overline{M_\mu}$$ Lorentz invariant

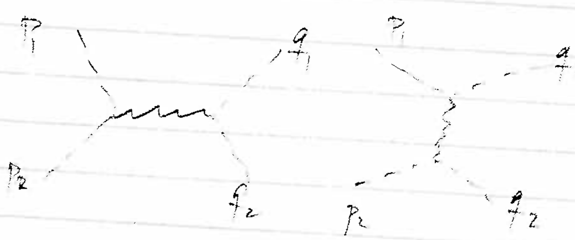
Electromagnetic pion scattering

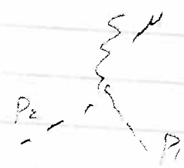
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \partial_\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi$$
$$- ie A_\mu \left[\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi \right] - e^2 A_\mu^2 \phi^* \phi$$

 $\frac{1}{i(2\pi)^4} \frac{1}{p^2+m^2}$

 $\frac{1}{i(2\pi)^4} \frac{1}{k^2} \left(\delta_{\mu\nu} - (1-\lambda^{-2}) \frac{k_\mu k_\nu}{k^2} \right)$

$\pi^+\pi^- \to \pi^+\pi^-$





$i(2\pi)^4 \delta^4(\ldots)(-ie) \times (ip_1 + ip_2)_\mu$

$i(2\pi)^4 \delta^4(\ldots)(-e^2)\delta_{\mu\nu}$

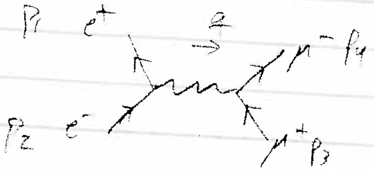
$$\mathcal{M} = e^2 \left\{ (p_1+p_2)_\mu \frac{1}{(p_1-p_2)^2} \left(\delta_{\mu\nu} - (1-\lambda^{-2}) \frac{(p_1-p_2)_\mu (q_2-q_1)_\nu}{(p_1-p_2)^2} \right) (-q_1-q_2)_\nu \right.$$
$$\left. + (p_1-q_1)_\mu \frac{1}{(p_1+q_1)^2} \left(\delta_{\mu\nu} - (1-\lambda^{-2}) \frac{(p_1+q_1)_\mu (p_2+q_2)_\nu}{(p_1+q_1)^2} \right) (p_2-q_2)_\nu \right\}$$
$$= e^2 \left(\frac{u-s}{t} + \frac{u-t}{s} \right) \qquad e^2 = 4\pi\alpha$$

$=$

A theory w/ massless vector particles must have gauge invariance or the theory is not Lorentz invariant. Theories with massive vector particles need gauge invariance or they are not renormalizable, they also need ghosts to be unitary. Ghosts correct unitarity violations brought about by the vector boson propagator. Proving this is what Gauge theory is all about.

See Diagrammatica by M. Veltman 0-521-45692-4 let's talk QED (spin $\frac{1}{2}$ particles)

$e^+e^- \to \mu^+\mu^-$ massless $\frac{m_e}{m_\mu} \sim \frac{1}{200}$



$$iM = \bar{v}^s(p_1)(-ie\gamma^\mu)u^s(p_2)\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)\bar{u}^r(p_4)(-ie\gamma^\nu)v^r(p_3)$$

$(\bar{v}\gamma^\mu u)^* = \bar{u}\gamma^\mu v$

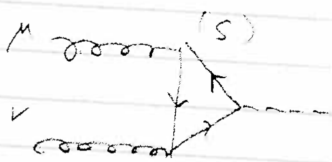
$$= \frac{ie^2}{q^2}\left(\bar{v}(p_1)\gamma^\mu u(p_2)\right)\left(\bar{u}(p_4)\gamma_\mu v(p_3)\right)$$

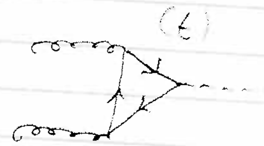
$$|M|^2 = \frac{e^4}{q^4}\left(\bar{v}\gamma^\mu u\bar{u}\gamma^\nu v\right)\left(\bar{u}\gamma_\mu v\bar{v}\gamma_\nu u\right)$$

this goes off course

What about QCD (non-abelian gauge theory)

let's find $\hat{\sigma}(gg \to h^0)$





$M = M_s + M_t$

$M = \epsilon_\mu \epsilon_\nu M^{\mu\nu}$

fermion loop

$$M_s = (ig\gamma_\mu T^a_{jk})\left(\frac{i((\not{k}-\not{p}_1)+m_t)}{(k-p_1)^2-m_t^2}\right)\left(\frac{m}{v}\right)\left(\frac{i(\not{k}+\not{p}_2)+m_t}{(k+p_2)^2-m_t^2}\right)\left(ig\gamma_\nu T^a_{ik}\right)\left(\frac{i(\not{k})+m_t}{k^2-m_t}\right)(-1)$$

$$= -g^2(T^a_{jk}T^a_{ki})\frac{m}{v}\int\frac{d^4k}{(2\pi)^4}\frac{Tr\left[\gamma_\mu(\not{k}-\not{p}_1)+m)((\not{k}+\not{p}_2)+m)\gamma_\nu(\not{k}+m)\right]}{[k^2-m^2][(k-p_1)^2-m^2][(k+p_2)^2-m^2]}$$

$(p_1,\mu) \leftrightarrow (p_2,\nu)$ $M = 2M_s$

$$Tr\left[(\not{k}+m)\gamma_\mu(\not{k}-\not{p}_1+m)(\not{k}+\not{p}_2+m)\gamma_\nu\right] = 4m\Big[4k_\mu k_\nu + 2(k_\mu p_{1\nu} - k_\nu p_{2\mu})$$
$$-(p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})$$
$$+\eta_{\mu\nu}(m^2-k^2-p_1\cdot p_2)\Big] = I_{\mu\nu}$$

finite for $d=4$ $\Gamma(3-\frac{n}{2})$

$$i\mathcal{M} = 4g^2 (T^a T^a) \frac{m}{v} \int_0^1 dx \int_0^{1-x} dy \, \frac{d^n k}{(2\pi)^n} \frac{I_{\mu\nu}}{(k^2 + 2kQ - m^2)^3} \qquad Q = yP_2 - xP_1$$

$$\vdots$$

$$i\mathcal{M} = -\frac{\alpha_s m^2}{\pi v} \delta^{ab} \left(\eta^{\mu\nu} \frac{M_h^2}{2} - p^\nu q^\mu \right) \int dx \, dy \left(\frac{1 - 4xy}{m^2 - M_h^2 xy} \right) \epsilon_\mu \epsilon_\nu$$

A Great modern reference on all things loopy

A. Smirnov Evaluating Feynman Integrals (Springer) 3-540-23933-2
Feynman Integral Calculus (Springer) 3-540-30610-2

A Nice two-loop formalism http://www.arXiv.org/abs/hep-ph/0508242 Grozin.

Lectures on Multiloop Calculations Int J. Mod. Phys A19 473-520 (2004)