Noncommutative Geometry, Quantum Fields, and Motives: a bird eye view

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Lecture 1: Tuesday Jan 15, 2008

General information about the class:

The material covered in this class is mostly based on

- Alain Connes and Matilde Marcolli, *Noncommutative Geometry, Quantum Fields, and Motives*, Colloquium Publications, Vol.55, American Mathematical Society, 2008.

Other reading material will be distributed in class and listed on the course webpage, along with notes of the lectures.

Course webpage:

`http://www.math.fsu.edu/ marcolli/NCGcourse.html`

Other information:

Office hours: by appointment

Research Seminar

Perturbative renormalization in Quantum Field Theory

Action $S(A) = \int \mathcal{L}(A)\, d^D x$, Lagrangian $\mathcal{L}\,(A) = \frac{1}{2}(\partial A)^2 - \frac{m^2}{2} A^2 - \mathcal{L}_{\text{int}}(A)$

$$\langle \mathcal{O} \rangle = \mathcal{N} \int \mathcal{O}(A)\, e^{i \frac{S(A)}{\hbar}}\, [dA]$$

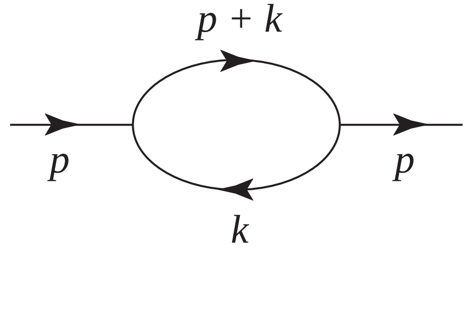
$$S_{eff}(A) = S_0(A) + \sum_{\Gamma \in 1PI} \frac{\Gamma(A)}{\sigma(\Gamma)}$$

$$\Gamma(A) = \frac{1}{N!} \int_{\sum p_j = 0} \widehat{A}(p_1) \ldots \widehat{A}(p_N)\, U(\Gamma(p_1, \ldots, p_N))\, dp_1 \ldots dp_N$$

Regularization and Renormalization

$$\int \frac{1}{k^2 + m^2} \frac{1}{((p+k)^2 + m^2)}\, d^D k$$

ϕ^3-theory $D = 4$ divergent



Hopf algebras and quantum field theory (Connes–Kreimer theory)

BPHZ renormalization:

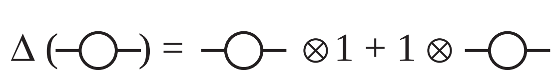
$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

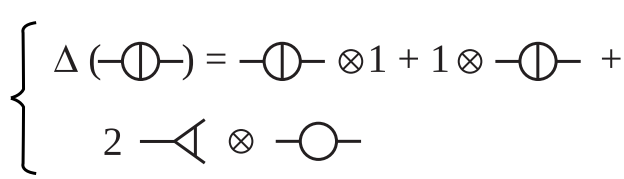
$$C(\Gamma) = -T(\overline{R}(\Gamma)) = -T\left(U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma) \right)$$

$$R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

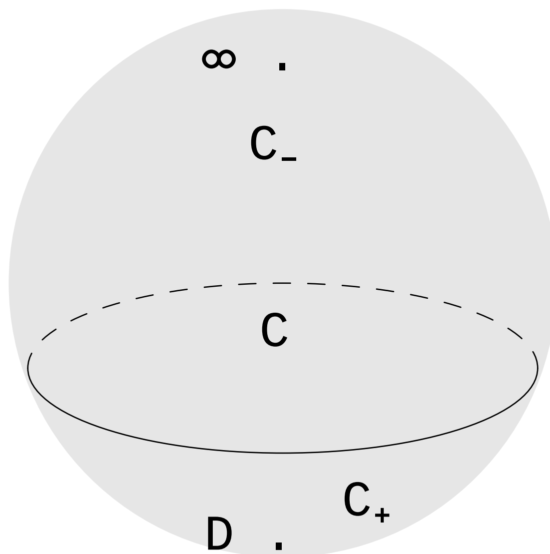
Hopf algebra of Feynman graphs:

$$\Delta(\Gamma) = \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$





The BPHZ renormalization and Birkhoff factorization of loops (CK)



$$\gamma(z) = \gamma_-(z)^{-1}\, \gamma_+(z),$$

$\Delta =$ small disk, $\Delta^* = \Delta \setminus \{0\}$, $z \in \Delta^*$

Dimensional regularization $z \in \mathbb{C}^*$

$$\gamma(z) \Leftrightarrow U(\Gamma), \quad \gamma_-(z) \Leftrightarrow C(\Gamma), \quad \gamma_+(z) \Leftrightarrow R(\Gamma)$$

Introducing motives

Cutting out pieces of algebraic varietes $h^i(X)$

$$\text{Hom}((X,p,m),(Y,q,n)) = q\,\text{Corr}^{m-n}_{/\sim}(X,Y)\,p$$

$p^2 = p$, $q^2 = q$, $\mathbb{Q}(m) =$ Tate motives

$$\omega : \mathcal{M}_{\mathbb{K}} \to Vect_{\mathbb{Q}} \quad X \mapsto H_B(X,\mathbb{Q})$$

Motivic Galois groups: (Tate $G = \mathbb{G}_m$)

Periods: $\int_X \omega$

Multiple zeta values $\overset{??}{\rightsquigarrow}$ Feynman integrals

Feynman motives and their periods (BEK)

Feynman trick:

$$\frac{1}{ab} = \int_0^1 \frac{dt}{(ta + (1-t)b)^2}$$

More generally: integral on a simplex

Feynman rules for graph $\Gamma \Rightarrow$

$$\int_\Sigma \frac{dv}{\Psi_\Gamma^2}$$

$\Psi_\Gamma =$ graph polynomial

Graph hypersurface:

$$X_\Gamma = \{t \in \mathbb{P}^N \,:\, \Psi_\Gamma(t) = 0\}$$

Cohomology of $\mathbb{P}^N \smallsetminus X_\Gamma$

Counterterms and beta function

Generator of renormalization group

$$\beta = \frac{d}{dt} F_t|_{t=0}$$

Counterterms reconstructed from the beta function ('t Hooft–Gross relations):

$$\gamma_-(z) = \mathsf{T} e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\, dt}$$

Time ordered exponential

$$\mathsf{T} e^{\int_a^b \alpha(t)\, dt} = 1 + \sum_1^\infty \int_{a \le s_1 \le \cdots \le s_n \le b} \alpha(s_1) \cdots \alpha(s_n) \prod ds_j$$

Renormalization and iterated integrals

Data of renormalization $\Rightarrow$ loops with

$$\gamma_\mu(z) = \mathrm{T}e^{-\frac{1}{z}\int_\infty^{-z\log\mu}\theta_{-t}(\beta)\,dt}\,\theta_{z\log\mu}(\gamma_{\mathrm{reg}}(z))$$

$$\gamma_{\mu^+}(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^{-z\log\mu}\theta_{-t}(\beta)\,dt}\,\theta_{z\log\mu}(\gamma_{\mathrm{reg}}(z))$$

$$\gamma_-(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^{\infty}\theta_{-t}(\beta)\,dt}$$

Time ordered exponential $\mathrm{T}e^{\int_a^b \alpha(t)\,dt} = g(b)$ solution of diff equation

$$dg(t) = g(t)\alpha(t)dt \qquad \text{with } g(a) = 1$$

Divergences of QFT $\Rightarrow$ Differential systems

Differential Galois theory

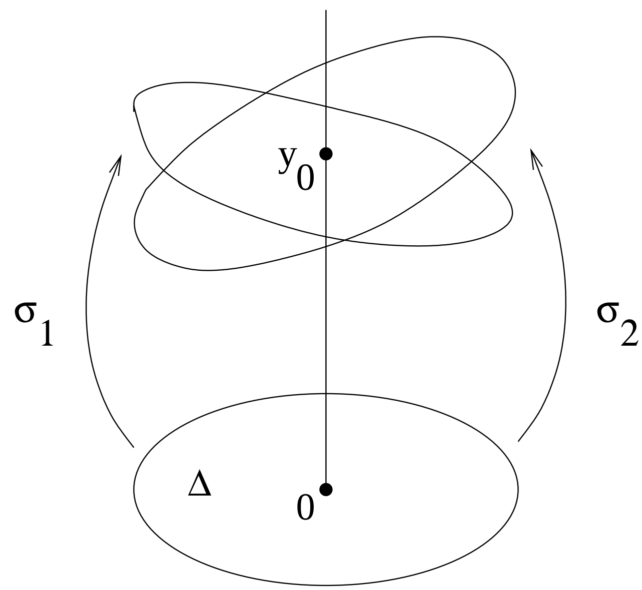
Hilbert 21st problem: Reconstruct differential equations from monodromy representation

$\Rightarrow$ Riemann–Hilbert correspondence: Classify differential systems with singularities by representations

Differential Galois group

Flat equisingular connections

Principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle $\mathbb{G}_m \to B \xrightarrow{\pi} \Delta$



Restrictions to different sections same type of singularity

Equisingular vector bundles $\Leftrightarrow$ $Rep_{\mathbb{U}^*}$

$\mathbb{U}^*$ and the renormalization group

Free graded Lie algebra $\mathcal{L}_\bullet = \mathcal{F}(1,2,3,\cdots)_\bullet$ generators e_{-n} deg $n > 0$

Hopf algebra $\mathcal{H}_u = \mathcal{U}(\mathcal{L}_\bullet)^\vee$ dual to $\mathcal{U}$

$$\mathcal{U}^* = \mathcal{U} \rtimes \mathbb{G}_m$$

In a given physical theory: generator $e_{-n} \mapsto \beta_n$

$$\beta = \sum_n \beta_n$$

n-loop component of the beta function

$$e = \sum_n e_{-n} \mapsto \beta$$

renormalization group as subgroup of $\mathbb{U}^*$

Renormalization and motives

- Periods of mixed Tate motives from Feynman integrals (Broadhurst–Kreimer)

- Graph hypersurfaces can be arbitrary motives (Belkale–Brosnan)

- Motives from Feynman integrals (Bloch–Esnault–Kreimer)

- Mixed Tate motives with

$$G = \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$$

(Deligne–Goncharov)

Noncommutative spaces

Equivalence relation $\mathcal{R}$ on X:
quotient $Y = X/\mathcal{R}$
Even for "good" X usually "bad" Y

Classical: functions on the quotient
$\mathcal{A}(Y) := \{f \in \mathcal{A}(X) \,|\, f$ is $\mathcal{R}-$invariant$\}$
$\Rightarrow$ often too few functions
$\mathcal{A}(Y) = \mathbb{C}$ only constants

NCG: $\mathcal{A}(Y)$ noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation

Convolution product

$$(f_1 * f_2)(x, y) = \sum_{x \sim u \sim y} f_1(x, u) f_2(u, y)$$

involution $f^*(x, y) = \overline{f(y, x)}$.

Spectral triples

Riemannian geometry: X with metric tensor g

$$(C^\infty(X), L^2(X,S), \not{D})$$

Dirac operator and spinors

Noncommutative Riemannian geometries

$$(\mathcal{A}, \mathcal{H}, \mathcal{D})$$

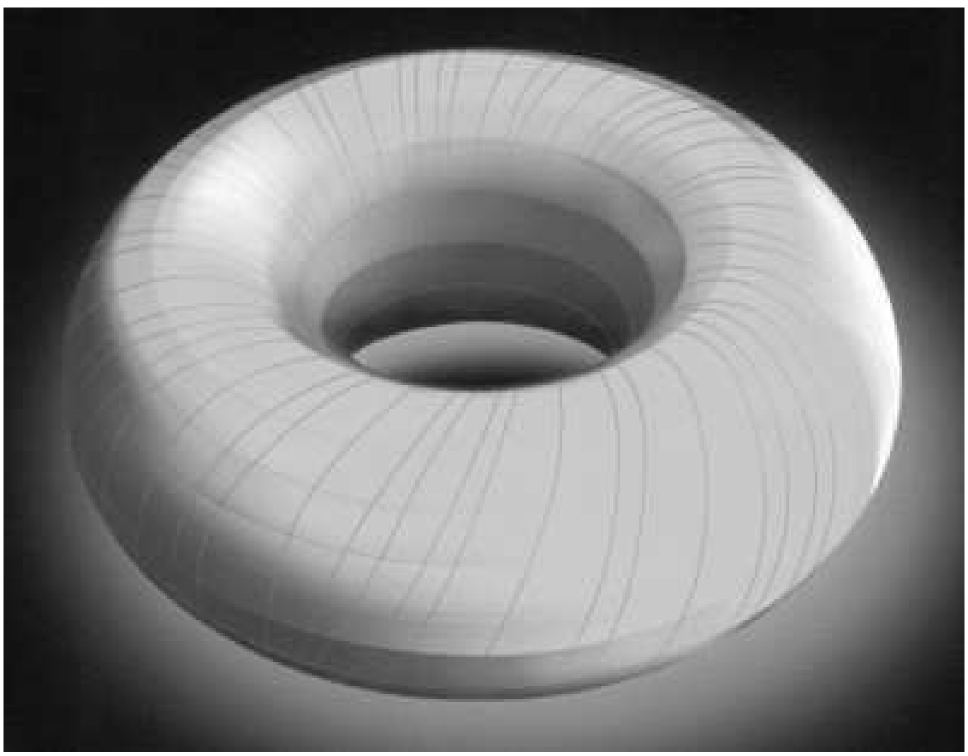
NC algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$

Unbounded operator $\mathcal{D}$ with $\mathcal{D}^* = \mathcal{D}$

$$[\mathcal{D}, a] \quad \text{bounded} \quad \forall a \in \mathcal{A}_\infty \subset \mathcal{A}$$

Examples of noncommutative spaces

Noncommutative tori: $S^1/\mathbb{Z}$ irrational rotation



Elliptic curves $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ with $|q| < 1$

Degeneration for $q \to e^{2\pi i\theta} \in S^1$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$

The spectral action

Action functional for spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$

$$\mathrm{Tr}\,(f(\mathcal{D}/\Lambda))$$

Λ mass scale, $f > 0$ even function

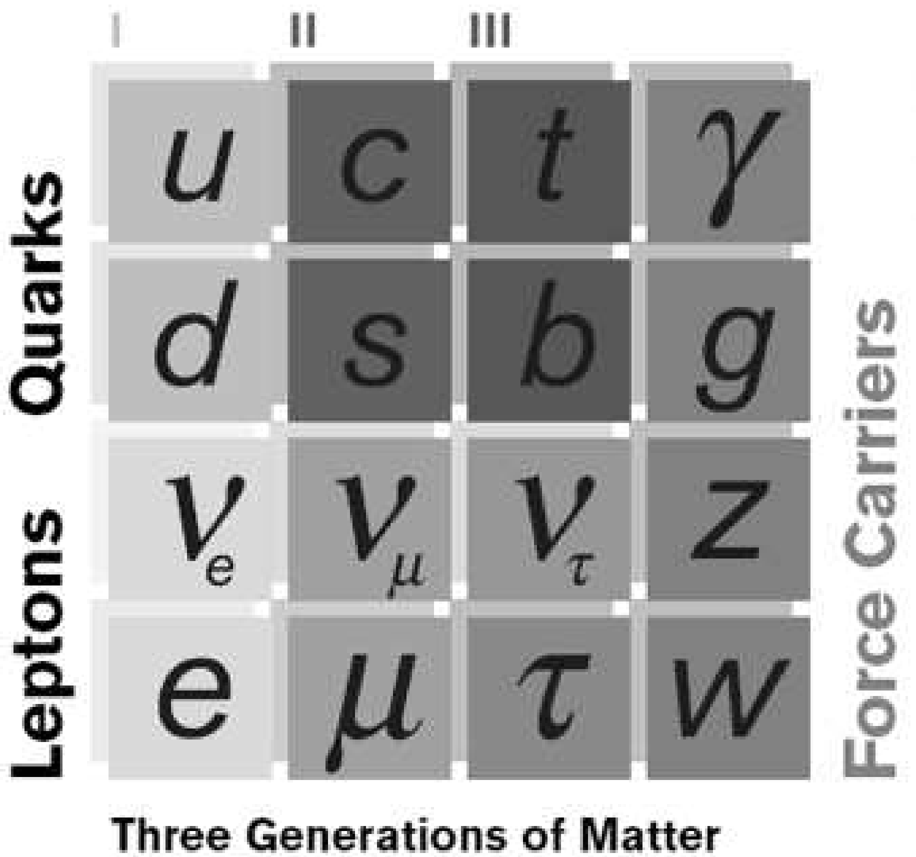
Asymptotic expansion

$$\mathrm{Tr}\,(f(D/\Lambda)) \sim \sum_k f_k\,\Lambda^k {\int\!\!\!\!\!-} |D|^{-k} + f(0)\,\zeta_D(0) + o(1)$$

with $f_k = \int_0^\infty f(v)\, v^{k-1}\, dv$

Contributions from $k \in$ Dimension Spectrum

The Standard Model of elementary particle physics



- coupling with gravity $S_{EH} + S_{SM}$

- neutrino mixing

The problem: Standard Model Lagrangian

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu -$$
$$\partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu -$$
$$igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) +$$
$$Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) -$$
$$A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) -$$
$$\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu -$$
$$Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) +$$
$$g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H -$$
$$2M^2 \alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 -$$
$$\beta_h \left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h -$$
$$g\alpha_h M \left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) -$$
$$\frac{1}{8}g^2\alpha_h \left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2\right) -$$
$$gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H -$$
$$\frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) +$$
$$\frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) +$$
$$\frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + M\,(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) -$$
$$ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) -$$
$$ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) -$$
$$\frac{1}{4}g^2 W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) -$$
$$\frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right) - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) + \frac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- -$$
$$g^2 s_w^2 A_\mu A_\mu\phi^+\phi^- + \frac{1}{2}ig_s\,\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial +$$
$$m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j +$$
$$igs_w A_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right) + \frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1 +$$

$$
\begin{gathered}
\gamma^5)\nu^\lambda)+(\bar{e}^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda)+(\bar{d}_j^\lambda\gamma^\mu(\tfrac{4}{3}s_w^2-1-\gamma^5)d_j^\lambda)+(\bar{u}_j^\lambda\gamma^\mu(1-\tfrac{8}{3}s_w^2+ \\
\gamma^5)u_j^\lambda)\} + \frac{ig}{2\sqrt{2}}W_\mu^+\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5){U^{lep}}_{\lambda\kappa}e^\kappa) + (\bar{u}_j^\lambda\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d_j^\kappa)\right) + \\
\frac{ig}{2\sqrt{2}}W_\mu^-\left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}_j^\kappa C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u_j^\lambda)\right) + \\
\frac{ig}{2M\sqrt{2}}\phi^+\left(-m_e^\kappa(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m_\nu^\lambda(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1+\gamma^5)e^\kappa\right) + \\
\frac{ig}{2M\sqrt{2}}\phi^-\left(m_e^\lambda(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m_\nu^\kappa(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right) - \\
\frac{g}{2}\frac{m_\nu^\lambda}{M}H(\bar{\nu}^\lambda\nu^\lambda) - \frac{g}{2}\frac{m_e^\lambda}{M}H(\bar{e}^\lambda e^\lambda) + \frac{ig}{2}\frac{m_\nu^\lambda}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda) - \frac{ig}{2}\frac{m_e^\lambda}{M}\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda) - \\
\frac{1}{4}\,\bar{\nu}_\lambda\, M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa - \frac{1}{4}\,\overline{\bar{\nu}_\lambda\, M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa} + \\
\frac{ig}{2M\sqrt{2}}\phi^+\left(-m_d^\kappa(\bar{u}_j^\lambda C_{\lambda\kappa}(1-\gamma^5)d_j^\kappa) + m_u^\lambda(\bar{u}_j^\lambda C_{\lambda\kappa}(1+\gamma^5)d_j^\kappa\right) + \\
\frac{ig}{2M\sqrt{2}}\phi^-\left(m_d^\lambda(\bar{d}_j^\lambda C^\dagger_{\lambda\kappa}(1+\gamma^5)u_j^\kappa) - m_u^\kappa(\bar{d}_j^\lambda C^\dagger_{\lambda\kappa}(1-\gamma^5)u_j^\kappa\right) - \\
\frac{g}{2}\frac{m_u^\lambda}{M}H(\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2}\frac{m_d^\lambda}{M}H(\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2}\frac{m_u^\lambda}{M}\phi^0(\bar{u}_j^\lambda\gamma^5 u_j^\lambda) - \frac{ig}{2}\frac{m_d^\lambda}{M}\phi^0(\bar{d}_j^\lambda\gamma^5 d_j^\lambda) + \\
\bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu\bar{G}^a G^b g_\mu^c + \bar{X}^+(\partial^2-M^2)X^+ + \bar{X}^-(\partial^2-M^2)X^- + \\
\bar{X}^0(\partial^2-\frac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_wW_\mu^+(\partial_\mu\bar{X}^0X^- - \partial_\mu\bar{X}^+X^0) + \\
igs_wW_\mu^+(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+Y) + igc_wW_\mu^-(\partial_\mu\bar{X}^-X^0 - \partial_\mu\bar{X}^0X^+) + \\
igs_wW_\mu^-(\partial_\mu\bar{X}^-Y - \partial_\mu\bar{Y}X^+) + igc_wZ_\mu^0(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) + \\
igs_wA_\mu(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) - \\
\frac{1}{2}gM\left(\bar{X}^+X^+H + \bar{X}^-X^-H + \frac{1}{c_w^2}\bar{X}^0X^0H\right) + \\
\frac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+X^0\phi^+ - \bar{X}^-X^0\phi^-\right) + \frac{1}{2c_w}igM\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + \\
igMs_w\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + \frac{1}{2}igM\left(\bar{X}^+X^+\phi^0 - \bar{X}^-X^-\phi^0\right)\ .
\end{gathered}
$$

NCG models of particle physics

Minimal mathematical input $\Rightarrow$ SM Lagrangian *derived* by calculation

Classification of finite geometries

• Product of spacetime by "finite NC space"

• Real structure on a spectral triple

• Finite space: metric dimension zero but "homological" dimension six

• All possible Dirac operators on the finite space $\Rightarrow$ physical properties (color unbroken, values of hypercharges, etc)

The Yukawa parameters of the Standard Model

- Cabibbo–Kobayashi–Maskawa matrix (quark masses, mixing angles, phase)

- Pontecorvo–Maki–Nagakawa–Sakata matrix (lepton masses, including neutrinos, mixing angles and phase)

- Majorana mass terms for neutrinos

Moduli space of Dirac operators:

$$\mathcal{C}_1 \times \mathcal{C}_3$$

lepton and quark sectors

$$\mathcal{C}_3 = (K \times K)\backslash(G \times G)/K$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection fiber symm matrices mod $M_R \mapsto \lambda^2 M_R$; $\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$

Bosonic and fermionic parts of the action

Bosonic part from asymptotic formula for the spectral action:

- Cosmological terms
- Riemann curvature terms
- Higgs minimal coupling and quartic potential
- Higgs mass terms
- Yang–Mills terms for gauge bosons

Fermionic part: from real structure, Pfaffian (Grassman fields)

- Fermion-Higgs coupling
- Gauge-fermion coupling
- Fermion doubling
- see-saw mechanism for neutrino masses

Physical predictions

- As in grand-unified theories:

$$\frac{g_3^2\, f_0}{2\pi^2} = \frac{1}{4}, \qquad g_3^2 = g_2^2 = \frac{5}{3}\, g_1^2$$

- Mass relation at unification

$$\sum_\sigma (m_\nu^\sigma)^2 + (m_e^\sigma)^2 + 3\,(m_u^\sigma)^2 + 3\,(m_d^\sigma)^2 = 8\, M^2$$

$M = W - mass$

- From mass relation and RGE ⇒ top quark mass estimate

- Higgs mass (168 GeV)

Dimensional regularization as a noncommutative geometry

$$\int e^{-\lambda\, q^2}\, d^D q = \pi^{D/2}\, \lambda^{-D/2}$$

NCG space X_z of Dimension Spectrum $z \in \mathbb{C}$

Dimensional Regularization: cup product of spectral triples

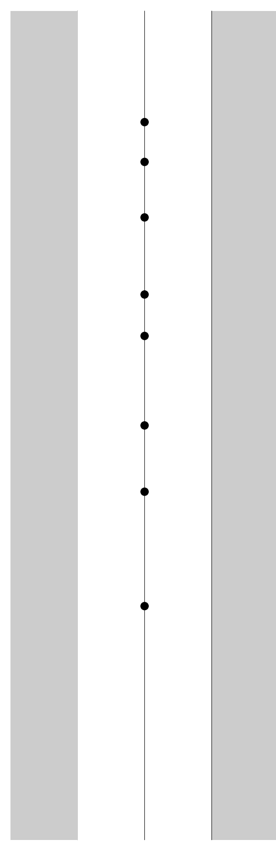
$$X \cup X_z$$

$X = (\mathcal{A}, \mathcal{H}, \mathcal{D})$ space time and finite space

$\Rightarrow$ Anomalies computations

The Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}, \quad \Re(s) > 1$$



Riemann Hypothesis:
zeros on the line $\Re(z) = 1/2$

Quantum mechanics of Riemann's zeta

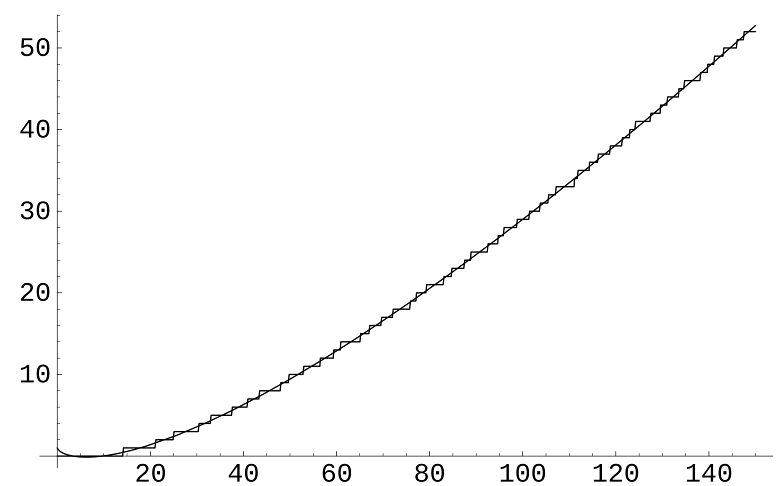
$$N(E) = \#\{\rho \,|\, \zeta(\rho) = 0, \quad \text{and} \quad 0 < \Im(\rho) \leq E\}$$

$$N(E) = \langle N(E) \rangle + N_{\text{osc}}(E)$$

$$\langle N(E) \rangle = \frac{E}{2\pi}\left(\log \frac{E}{2\pi} - 1\right) + \frac{7}{8} + o(1)$$

$$N_{\text{osc}}(E) = \frac{1}{\pi} \Im \log \zeta \left(\frac{1}{2} + iE\right)$$

$$\sim \frac{1}{\pi} \sum_{\gamma_p} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2 \sinh\left(\frac{m\lambda_p}{2}\right)} \sin(m\, E\, T_\gamma^\#)$$



Energy levels of a quantum mechanical system (absorption spectrum)

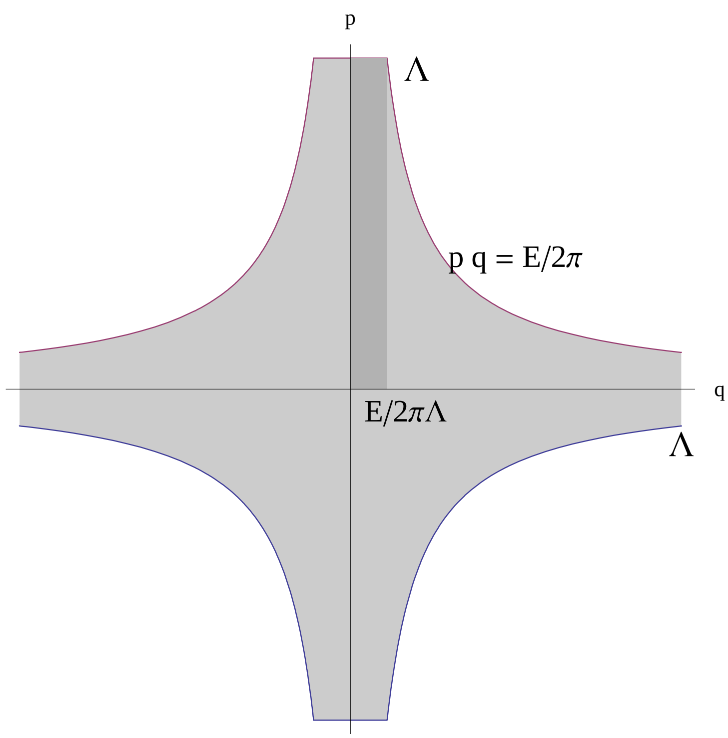
The scaling Hamiltonian (classical level)

$$H(q,p) = 2\pi qp$$

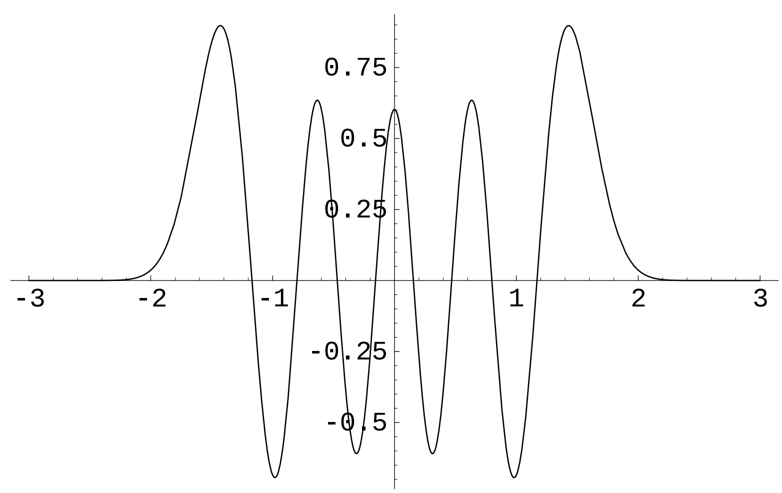
$$B = \{(q,p)\,:\ |H(q,p)| \le E\,, \quad |q| \le \Lambda\,, \quad |p| \le \Lambda\}$$

in $\mathbb{R}^2$ mod $(p,q) \mapsto (-p,-q)$

$$\mathrm{Vol}(B_+) = \frac{E}{2\pi} \times 2\log\Lambda - \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right)$$



Prolate spheroidal wave functions
(quantum)



Cutoffs in q and in p (cutoffs on function and its Fourier transform)

The adèle class space

$$\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z}$$

finite adèles

$$\mathbb{A}_{\mathbb{Q},f} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

full adèles

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f} \times \mathbb{R}$$

idèles classes

$$C_{\mathbb{Q}} = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})/\mathrm{GL}_1(\mathbb{Q}) = \hat{\mathbb{Z}}^* \times \mathbb{R}^*_+$$

adèle classes

$$X_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}/\mathrm{GL}_1(\mathbb{Q})$$

"bad" equivalence relation
$\Rightarrow$ Noncommutative space

Weil's explicit formula

Primes and zeros of zeta:

$$\pi(x) = \#\{p \text{ prime } \mid p \leq x\}$$

$$\pi'(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) + \int_x^\infty \frac{dt}{t(t^2-1)\log t} + \log \xi(0)$$

$$\mathrm{Li}(x) = \int_0^x \frac{du}{\log(u)} \sim \sum_k (k-1)! \frac{x}{\log(x)^k}$$

$$\xi(t) = \frac{s(s-1)}{2} \Gamma(s/2) \pi^{-s/2} \zeta(s)$$

A distributional form (Weil)

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\chi \in \widehat{C_{\mathbb{K},1}}} \sum_{Z_{\tilde{\chi}}} \widehat{h}(\tilde{\chi}, \rho) = \sum_v \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u,$$

Connes' spectral realization

$$0 \to L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*)_0 \to L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*) \to \mathbb{C}^2 \to 0$$

$f(0) = 0$ and $\hat{f}(0) = 0$

$$0 \to L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*)_0 \xrightarrow{\mathfrak{E}} L^2_\delta(C_\mathbb{Q}) \to \mathcal{H} \to 0$$

$$\mathfrak{E}(f)(g) = |g|^{1/2} \sum_{q \in \mathbb{Q}^*} f(qg), \quad \forall g \in C_\mathbb{Q}$$

$$U(h) = \int_{C_\mathbb{Q}} h(g)\, U_g\, d^*g \quad h \in \mathcal{S}(C_\mathbb{Q})$$

$$\mathcal{H} = \oplus_\chi \mathcal{H}_\chi \quad \chi \in \text{characters of } \hat{\mathbb{Z}}^*$$

$\mathcal{H}_\chi = \{\xi \in \mathcal{H} : U_g \xi = \chi(g)\xi\}$ w/ $\mathbb{R}^*_+$-action gen by D_χ

$$\mathrm{Spec}(D_\chi) = \left\{ s \in i\mathbb{R} \,|\, L_\chi\left(\frac{1}{2} + s\right) = 0 \right\}$$

Trace formula

$$\mathrm{Tr}(R_\Lambda U(h)) = 2h(1)\log\Lambda + \sum_{v \in S} \int'_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1-u|} d^*u + o(1)$$

Periodic orbits of the action of $C_\mathbb{Q}$ on $X_\mathbb{Q} \smallsetminus C_\mathbb{Q}$

L-functions of algebraic varieties

Complete zeta function $\Gamma(s/2)\pi^{-s/2}\zeta(s)$

$$\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\} = \{2, 3, 5, 7, 11, \dots, \infty\}$$

"infinite prime": embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$

X algebraic over $\mathbb{K}$

$$L(H^m(X), s) = \prod_v L_v(H^m(X), s)$$

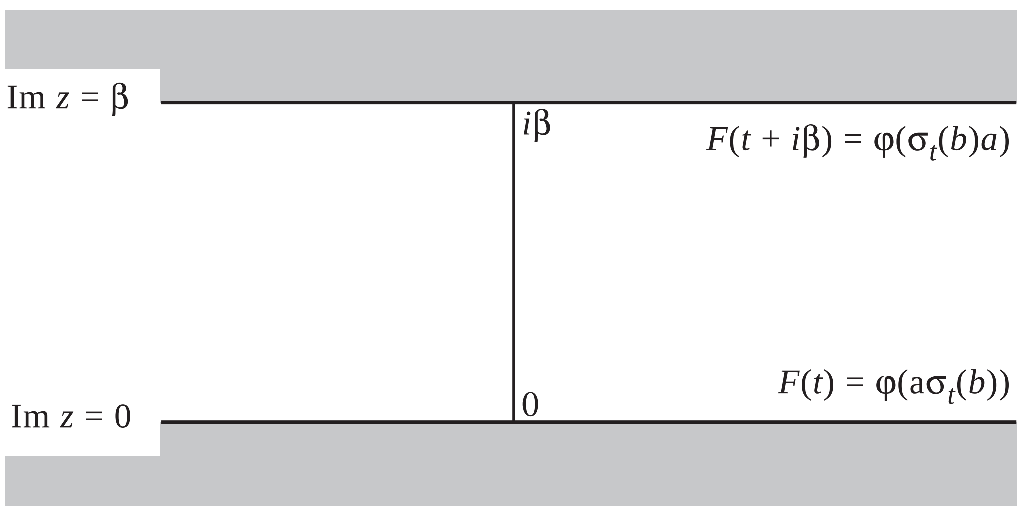
$$L_v(H^m(X), s) = \det\left(1 - Fr_v^* N(v)^{-s} | H^m(\bar{X}, \mathbb{Q}_\ell)^{I_v}\right)^{-1}$$

$$L(X, s) = \prod_{i=0}^{n} L(H^i(X), s)^{(-1)^{i+1}}$$

$$L(H^*, s) = \begin{cases} \prod_{p,q} \Gamma_{\mathbb{C}}(s - \min(p, q))^{h^{p,q}} \\ \prod_{p<q} \Gamma_{\mathbb{C}}(s - p)^{h^{p,q}} \prod_p \Gamma_{\mathbb{R}}(s - p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s - p + 1)^{h^{p,-}} \end{cases}$$

Trace formula for archimedean local factors

Quantum Statistical Mechanics $(\mathcal{A}, \sigma_t)$



Equilibrium states: $\varphi(a) = \frac{1}{Z(\beta)} \mathrm{Tr}(a e^{-\beta H})$

KMS condition generalizes Gibbs states

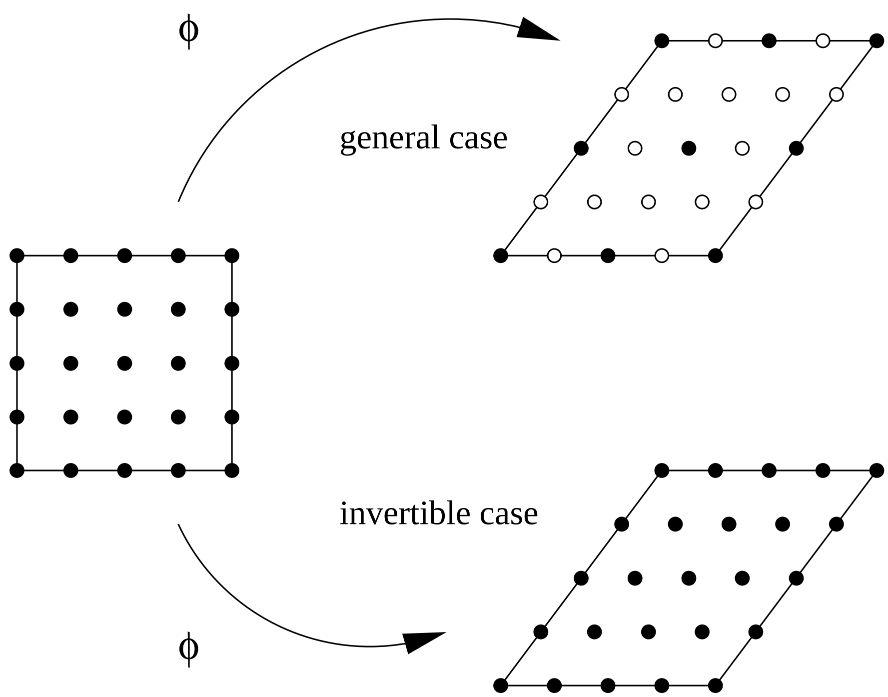
Points of NC spaces = low temperature KMS

$\mathbb{Q}$-lattices

(Λ, ϕ) $\mathbb{Q}$-lattice in $\mathbb{R}^n$

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

group *homomorphism* (invertible $\mathbb{Q}$-lat if isom)



Commensurability: $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$
iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

The Bost–Connes system

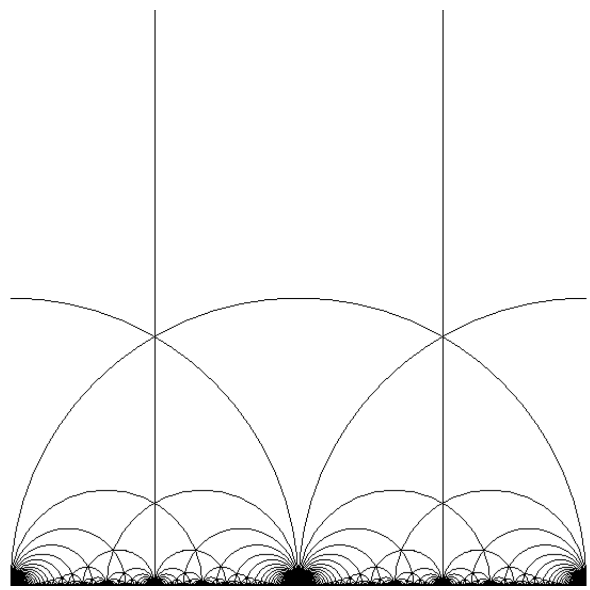
Quantum statistical mechanics	Quantum field theory
Commensurability classes of $\mathbb{Q}$-lattices modulo scaling	Commensurability classes of $\mathbb{Q}$-lattices
$\mathcal{A} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times$	$\mathcal{A} \rtimes_{\sigma_t} \mathbb{R}$
Time evolution σ_t	Energy scaling $U(\lambda)$, $\lambda \in \mathbb{R}^*_+$
$\{\log p\}$ as frequencies	$\{\log p\}$ as periods of orbits
Arithmetic rescaling μ_n	Renormalization group $\mu\partial_\mu$
Symmetry group $\hat{\mathbb{Z}}^*$ as Galois action on $T = 0$ states	Idèle class group as gauge group
System at zero-temperature	$\mathrm{GL}_1(\mathbb{Q})\backslash\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$
System at critical temperature (Riemann's ζ as partition function)	Spectral realization (Zeros of ζ absorption spectrum)
Type III_1	Type II_∞

Modular curves and statistical mechanics

• 2-dimensional $\mathbb{Q}$-lattices

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}\tau), \rho)$$

$\lambda \in \mathbb{C}^*$, $\tau \in \mathbb{H}$, $\rho \in M_2(\hat{\mathbb{Z}})$



$$\mathrm{GL}_2^+(\mathbb{Q}) \backslash (M_2(\mathbb{A}_{\mathbb{Q},f}) \times \mathrm{GL}_2(\mathbb{R}))$$

• Noncommutative compactification of modular curves: $C(\mathbb{P}^1(\mathbb{R})) \rtimes \mathrm{SL}_2(\mathbb{Z})$

Hilbert's 12th problem and QFT

Hilbert 12th problem: explicit generators of maximal abelian extension $\mathbb{K}^{ab}$ and Galois action $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$

System	GL_1	GL_2	$\mathbb{K} = \mathbb{Q}(\sqrt{-d})$
Partition function	$\zeta(\beta)$	$\zeta(\beta)\zeta(\beta-1)$	$\zeta_{\mathbb{K}}(\beta)$
Symmetries	$\mathbb{A}^*_{\mathbb{Q},f}/\mathbb{Q}^*$	$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})/\mathbb{Q}^*$	$\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$
Automorphisms	$\hat{\mathbb{Z}}^*$	$\mathrm{GL}_2(\hat{\mathbb{Z}})$	$\hat{\mathcal{O}}^*/\mathcal{O}^*$
Endomorphisms		$\mathrm{GL}_2^+(\mathbb{Q})$	$\mathrm{Cl}(\mathcal{O})$
Galois group	$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$	$\mathrm{Aut}(F)$	$\mathrm{Gal}(\mathbb{K}^{\mathrm{ab}}/\mathbb{K})$
Extremal KMS_∞	$Sh(\mathrm{GL}_1, \pm 1)$	$Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$	$\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$

$$\gamma\, \varphi(x) = \varphi(\theta(\gamma)\, x)$$

Noncommutative motives (endomotives)

$\mathcal{A}_{\mathbb{K}} = A \rtimes S$

$$A = \varinjlim_{\alpha} A_\alpha \quad X_\alpha = \mathrm{Spec}(A_\alpha)$$

$X_\alpha =$ Artin motives over $\mathbb{K}$
$S =$ unital abelian semigroup of endomorphisms

Correspondences as morphisms

Data: (X, S) endomotive $/\mathbb{K}$:

- C^*-algebra $\mathcal{A} = C(\mathcal{X}) \rtimes S$
- arithmetic subalgebra $\mathcal{A}_{\mathbb{K}} = A \rtimes S$
- state $\varphi : \mathcal{A} \to \mathbb{C}$ (from uniform μ measure)
- Galois action $G \subset \mathrm{Aut}(\mathcal{A})$

Quantum statistical mechanics
$\Rightarrow$ time evolution, KMS

Cooling, Distillation, and other alchemies

Classical points of NC space: low temperature KMS

$$\tilde{\Omega}_\beta = \Omega_\beta \times \mathbb{R}_+^*$$

$$\lambda(\epsilon, H) = (\epsilon, H + \log \lambda)$$

Dual system $(\hat{\mathcal{A}}, \theta)$ of $(\mathcal{A}, \sigma)$: $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$

$$\theta_\lambda(\int x(t) U_t \, dt) = \int \lambda^{it} x(t) U_t \, dt$$

• Restriction map: $\delta = (\mathrm{Tr} \circ \pi)^\natural : \hat{\mathcal{A}}_\beta \to C(\tilde{\Omega}_\beta)$
Inclusion of classical points

• Cokernel $D(\mathcal{A}, \varphi) = \mathrm{Coker}(\delta)$ (as "motive")

$$HC^*(D(\mathcal{A}, \varphi))$$

Connes' trace formula revisited

Restriction map δ for BC system $(C(\hat{\mathbb{Z}})\rtimes\mathbb{N},\sigma_t)$:

$$\delta(f)=\sum_{n\in\mathbb{N}} f(1,n\rho,n\lambda)=\mathfrak{E}(f)$$

Weil's explicit formula

$$\mathrm{Tr}(\vartheta(f)|_{\mathcal{H}^1})=\hat{f}(0)+\hat{f}(1)-\Delta\bullet\Delta f(1)-\sum_v \int'_{(\mathbb{K}_v^*,e_{K_v})}\frac{h(u^{-1})}{|1-u|}d^*u$$

• Connes' trace formula:
$\mathrm{Tr}(R_\Lambda U(f))$: only zeros on critical line
RH $\Leftrightarrow$ Trace formula (global)

• Cohomological version:
$\mathrm{Tr}(\vartheta(f)|_{\mathcal{H}^1})$: all zeros involved
RH $\Leftrightarrow$ positivity of trace of correspondences

$\Rightarrow$ Better for comparing with Weil's proof for function fields

The Weil proof and the adèles class space

$\mathbb{K} = \mathbb{F}_q(C)$ algebraic curve C

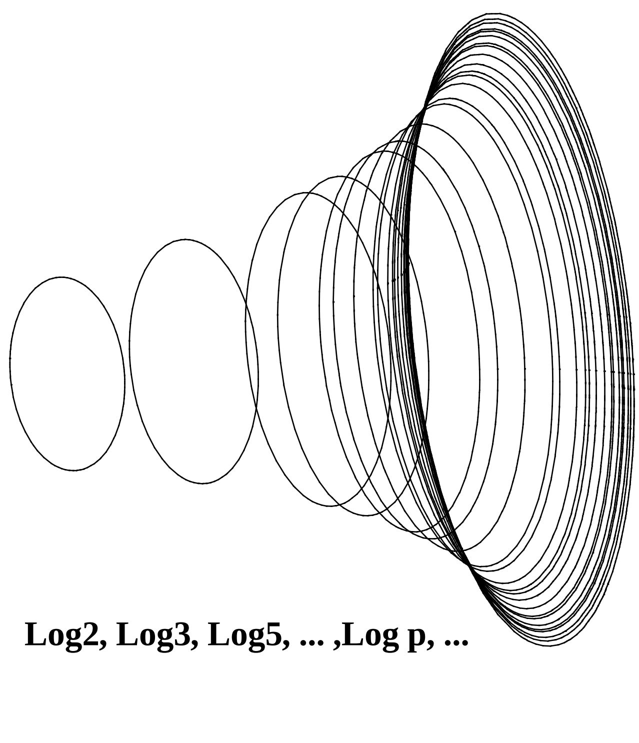
- Frobenius correspondence
- adjust degree by trivial correspondences
- Riemann–Roch: linear equivalent to effective
- positivity of Tr$(Z * \bar{Z})$

Alg Geom/NT	**NCG**
$C(\bar{\mathbb{F}}_q)$ alg points	$\Xi_{\mathbb{K}}$ classical points
Weil explicit formula	Tr$(\vartheta(f)\|_{\mathcal{H}^1})$
Frobenius correspondence	$Z(f) = \int_{C_{\mathbb{K}}} f(g)\, Z_g\, d^*g$
Trivial correspondences	$\mathcal{V} =$ Range(Tr $\circ\, \pi$)
Adjusting the degree by trivial correspondences	Fubini step on test functions in $\mathcal{V}$
Principal divisors	???
Riemann–Roch	Index theorem

Number Fields and Function Fields

- Frobenius and scaling

- Algebraic points of the curve and KMS states



Number Theory and Quantum Gravity

$\mathbb{Q}$-**lattices**	**Quantum Gravity**
$\mathbb{Q}$-lattice	Real spectral triple of dimension $d = 3$
Invertible $\mathbb{Q}$-lattice	Poincaré duality spectral triple
Commensurable pair of $\mathbb{Q}$-lattices	Spectral correspondence (or cobordism)
Scaling action	$D \mapsto \lambda D$
Composition of pairs	Composition of correspondences
$\gamma \mapsto \gamma^{-1}$	Contragredient correspondence
Groupoid C^*-algebra	Hecke algebra of functions of correspondences
Eisenstein series	$D \mapsto \mathrm{Tr}(D^{-n}) = \mathrm{Tr}(ds^n)$
Shimura variety	Moduli space of Dirac operators