Zeta functions hear the shape of Riemann surfaces

Gunther Cornelissen and Matilde Marcolli

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Compact Riemann surface X

$$X = \Gamma \backslash (\mathbb{P}^1(\mathbb{C}) - \Lambda_\Gamma)$$

Schottky uniformization

$\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ discrete purely loxodromic $\Gamma \simeq \mathbb{Z}^{*g}$

$\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ limit set

Γ-action on limit set Λ_Γ

Group completion and limit set

Y_g = Cayley graph of F_g Group completion $\bar{F}_g := \bar{Y}_g \smallsetminus Y_g$

$$\iota_\rho : \bar{F}_g \to \Lambda \qquad \lim_i w_i \mapsto \lim_i \rho(w_i)(x_0)$$

given point $x_0 \in \mathbb{P}^1(\mathbb{C})$ and embedding $\rho : F_g \hookrightarrow \mathrm{PGL}(2,\mathbb{C})$

reduced word w in the generators of F_g, $i(w)$ and $t(w)$ initial and terminal letters

$$w \subseteq v \text{ if } (\exists w_0)(v = w \cdot w_0) \text{ with } t(w) \neq i(w_0)^{-1}$$

$$\overrightarrow{w}_\rho := \{\iota_\rho(v) \ : \ v \in \bar{F}_g \text{ and } w \subseteq v\}$$

Commutative algebra $A = C(\Lambda)$

$A_\infty \subset A$ dense involutive subalgebra spanned by characteristic functions $\chi_{\overrightarrow{w}_\rho}$

$$A_\infty = C(\Lambda, \mathbb{Z}) \otimes \mathbb{C}$$

Patterson–Sullivan measure

Scaling by the Hausdorff dimension δ_H of Λ_Γ

$$(\gamma^* d\mu)(x) = |\gamma'(x)|^{\delta_H} \, d\mu(x), \quad \forall \gamma \in \Gamma$$

State $\tau : A_\infty \to \mathbb{C}$

$$\tau(\chi_{\overrightarrow{w}_\rho}) := \int_\Lambda \chi_{\overrightarrow{w}_\rho} \mathrm{d}\mu_\Lambda = \mu_\Lambda(\overrightarrow{w}_\rho).$$

$\tau(1) = 1 = \mu_\Lambda(\Lambda)$ and $\tau(a^* a) \geq 0$

GNS representation: inner product

$$\langle a | b \rangle := \tau(b^* a)$$

Spectral triples (Connes)

$\mathcal{S} = (A, H, D)$: C^*-algebra A represented in $\mathcal{B}(H)$
Hilbert space H
$A_\infty \subset A$ dense involutive subalgebra
self-adjiont operator D on H with compact resolvent

$$[D, a] \in \mathcal{B}(H) \quad \forall a \in A_\infty$$

Finite summability (p-summable)

$$\mathrm{Tr}(|D|^{-s}) < \infty \quad \forall s \geq p$$

Example: Riemannian spin manifolds

$$\mathcal{S} = (C^\infty(X) \subset C(X), L^2(X, S), \not{\partial}_X)$$

Zeta functions of spectral triples: $a \in A_\infty$

$$\zeta_{a,\mathcal{S}}(s) = \mathrm{Tr}(a|D|^s)$$

$\Re(s) << 0$

Can you hear the shape of a drum?

$\mathrm{Tr}(|\partial\!\!\!/_X|^s)$ not enough: isospectral manifolds
What about $\mathrm{Tr}(f|\partial\!\!\!/_X|^s)$?

Goal: Construct a (commutative) spectral triple encoding the action of Γ on Λ such that the family $\zeta_{a,\mathcal{S}}(s)$ determines the (anti)conformal class of the Riemann surface

Commutative spectral triple on $\Lambda = \Lambda_\Gamma$

$\mathcal{S}_X = (A, H, D)$

$A = C(\Lambda)$ with $A_\infty = C(\Lambda, \mathbb{Z}) \otimes \mathbb{C}$
$H =$ GNS representation for τ

Filtration: $A_\infty = \varinjlim A_n$ (reduced words length $\leq n$)

Dirac operator

$$D := \lambda_0 P_0 + \sum_{n \geq 1} \lambda_n (P_n - P_{n-1}),$$

$\lambda_n = (\dim A_n)^3$

$Q_n := P_n - P_{n-1}$ projection onto graded pieces: $H_n \ominus H_{n-1}$
words of exact length n

For $a \in A_n$ and $m \geq n$, a preserves A_m

$$[D, a] = \sum_{i=0}^{n} \lambda_i [Q_i, a]$$

finite sum: bounded

$$\mathrm{tr}((1+D^2)^{-1/2}) = 1 + \sum_{n=1}^{\infty} (1+\lambda_n^2)^{-1/2} (\dim H_n - \dim H_{n-1})$$

$$\leq 1 + \sum_{n=1}^{\infty} (1+\lambda_n^2)^{-1/2} \dim A_n$$

$$\leq 1 + \sum_{n=1}^{\infty} (\dim A_n)^{-2} \leq 1 + \sum_{n=1}^{\infty} (n+1)^{-2} \leq 2$$

with $\dim A_n \geq n+1 \Rightarrow$ 1-summable

Note: existence of a 1-summable triple and existence of a quasi-circle

Ends of words:

$$\overrightarrow{w_1} \cap \overrightarrow{w_2} = \overrightarrow{\max\{w_1, w_2\}}$$

$\max\{w, v\}$ largest if comparable in $\subseteq$ or $\emptyset$

Basis for H_n: χ_w for $|w| = n$

$$\langle \chi_w | \chi_v \rangle = \mu(\overrightarrow{\max\{v, w\}})$$

relation $\chi_{\overrightarrow{u}} = \sum_{\substack{|w|=n \\ u \subset w}} \chi_{\overrightarrow{w}}$

$$\dim A_n = \dim H_n = 2g(2g-1)^{n-1}$$

Orthonormalization: start with $|\Psi_e\rangle = \chi_\Lambda$ and

$$|\Psi_w\rangle := \frac{1}{\sqrt{\mu_X(\overrightarrow{w})}} \chi_{\overrightarrow{w}} \qquad (|w| = 1)$$

w length one $w \neq w_0$ chosen, then $\{|\Psi_w\rangle\}_{w \in I_1}$ with $I_1 := S \cup \{e\}$ on basis for H_1

Inductively $I_{n+1} = I_n \cup \bigcup_{|w|=n} V_w$ with $|w| = n$ and V_w set of $2g-2$ letters $\neq t(w)^{-1}$

$\Rightarrow \{\chi_{\overrightarrow{w}}\}_{w \in I_{n+1} - I_n}$ basis of $H_{n+1} \ominus H_n$

Zeta functions $\zeta_{a,\mathcal{S}_X}(s)$

$$\mathrm{tr}(aD^s) = 1 + \sum_w \langle \Psi_w | a \sum_{n \geq 1} \lambda_n^s (P_n - P_{n-1}) \Psi_w \rangle$$

$$= 1 + \sum_{n \geq 1} \lambda_n^s \, c_n(a)$$

$c_n(a) = \sum_{w \in I_n - I_{n-1}} \langle \Psi_w | a \Psi_w \rangle$

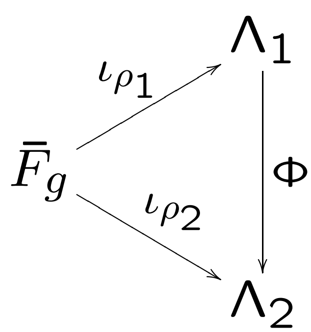
Lemma: Given X_1, X_2 compact Riemann surfaces $g \geq 2$

$$\zeta_{1,\mathcal{S}_{X_1}}(s) = \zeta_{1,\mathcal{S}_{X_2}}(s)$$

$\Rightarrow g_1 = g_2$ and

$$A_1 \cong A_2$$

C^*-algebra isomorphism from homeomorphism $\Phi : \Lambda_1 \to \Lambda_2$



Explicitly:

$$\zeta_{1,\mathcal{S}_X}(s) = 1 + \frac{2g-2}{2g-1} \cdot \frac{(2g)^{3s+1}}{1-(2g-1)^{3s+1}}$$

Computing $\zeta_{1,\mathcal{S}}(s)$:

$$\lambda_n = (\dim A_n)^3 = (2g)^3(2g-1)^{3n-3}$$

$$c_n(1) = \sum_{|w|\in I_n - I_{n-1}} \langle \Psi_w | \Psi_w \rangle$$

$$= \sum_{|w|\in I_n - I_{n-1}} 1 \;=\; 2g(2g-1)^{n-2}(2g-2)$$

$$\zeta_{1,\mathcal{S}}(s) = 1 + \sum_{n\geq 1} \lambda_n^s c_n(1) =$$

$$1 + (2g)^{3s+1}\,\frac{2g-2}{2g-1}\,\sum_{n\geq 1}(2g-1)^{(3s+1)(n-1)}$$

The condition $\zeta_{1,\mathcal{S}_{X_1}}(s) = \zeta_{1,\mathcal{S}_{X_2}}(s)$ gives

$$\frac{2g_1-2}{2g_1-1}\cdot\frac{2g_2-1}{2g_2-2}\cdot\left(\frac{g_1}{g_2}\right)^{3s+1} = \frac{1-(2g_1-1)^{3s+1}}{1-(2g_2-1)^{3s+1}}$$

for $\Re(s) << 0$. For $s \to -\infty$, rhs $\to 1$ and lhs$\to 0$ unless $g_1 = g_2$

Can then compare $\zeta_{a,\mathcal{S}_{X_1}}(s)$ and $\zeta_{a,\mathcal{S}_{X_2}}(s)$ for same $a \in A_1 \cong A_2$ (under above identification)

Lemma: $\zeta_{a,\mathcal{S}_{X_1}}(s) = \zeta_{a,\mathcal{S}_{X_2}}(s)$ gives

$$\sum_{n\geq 0} \Big(c_{n,1}(a) - c_{n,2}(a) \Big) \lambda_n^s \equiv 0$$

for $\Re(s) << 0$ gives Dirichlet series

$$\sum_{N\geq 0} \tilde{c}_N N^s \equiv 0$$

for $\Re(s) << 0$ with $\tilde{c}_N = c_{n,1}(a) - c_{n,2}(a)$ if $N = \lambda_n$ for some n, and $\tilde{c}_N = 0$ otherwise

$\Rightarrow \tilde{c}_N = 0$ for all N

$$c_{n,1}(a) = c_{n,2}(a)$$

Lemma: (inductively)

For $a = \chi_{\overrightarrow{\eta}}$ and w length $|w| = n < |\eta|$

$$\langle \Psi_w | a \Psi_w \rangle = \mu(\overrightarrow{\eta}) \cdot \kappa$$

κ depends on measures $\mu(\overrightarrow{v})$ words length $|v| < |\eta|$

Note: $c_{m-1}(a) \neq 0$ for $a = \chi_{\overrightarrow{\eta}}$ with $|\eta| = m$ since $\exists$ w supp Ψ_w intersects $\overrightarrow{\eta}$ and

$$c_{m-1}(a) = \sum_{w \in I_{m-1} - I_{m-2}} \langle \Psi_w | a \Psi_w \rangle \geq 0$$

hence $\kappa \neq 0$

Reconstruction of PS measure

Prop: $\zeta_{a,\mathcal{S}_{X_1}}(s) = \zeta_{a,\mathcal{S}_{X_2}}(s)$ gives

$$\mu_1(\overrightarrow{\eta}_{\rho_1}) = \mu_2(\overrightarrow{\eta}_{\rho_2})$$

for all $\eta \in F_g$, $\rho_i : F_g \to \Gamma_i \subset \mathrm{PGL}(2, \mathbb{C})$

$|\eta| = 0 \Rightarrow \overrightarrow{\eta}_{\rho_i} = \Lambda_i$ for $i = 1, 2$

$$c_{m-1,i}(\chi_{\overrightarrow{\eta}_\rho}) = \mu(\overrightarrow{\eta}_{\rho_i}) \cdot \kappa_i$$

$$c_{m-1,1}(\chi_{\overrightarrow{\eta}_{\rho_1}}) = c_{m-1,2}(\chi_{\overrightarrow{\eta}_{\rho_2}})$$

inductively: $\kappa_i = \kappa$ (shorter lengths) $\Rightarrow$

$$\mu(\overrightarrow{\eta}_{\rho_1}) = \mu(\overrightarrow{\eta}_{\rho_2})$$

Theorem $\zeta_{a,\mathcal{S}_{X_1}}(s) = \zeta_{a,\mathcal{S}_{X_2}}(s)$ for all $a \in A_\infty$ $\Rightarrow X_1$ and X_2 conformally or anti-conformally equivalent Riemann surfaces

Same genus from $a = 1$ hence $\rho_i : F_g \to \Gamma_i \subset$ PGL$(2, \mathbb{C})$ and isomorphism

$$\alpha = \rho_2 \circ \rho_1^{-1} : \Gamma_1 \stackrel{\simeq}{\to} \Gamma_2$$

$\Rightarrow \Phi : \Lambda_1 \to \Lambda_2$ homeomorphism

α-equivariant: $\Phi(\gamma \cdot x) = \alpha(\gamma)\Phi(x)$

Measure preserving: $\mu_2 \circ \Phi^* = \mu_1$ (from Prop)

$$\mu_2(\chi_{\Phi(\vec{w}_{\rho_1})}) = \mu_2(\chi_{\vec{w}_{\rho_2}}) = \mu_1(\chi_{\vec{w}_{\rho_1}})$$

Ergodic rigidity (Chengbo Yue)

Γ_1, Γ_2 geometrically finite subgroups of simple connected adjoint Lie groups G_1 and G_2 real rank one

Γ_1 Zariski dense in G_1

$\alpha : \Gamma_1 \to \Gamma_2$ be a type-preserving isomorphism

$\Rightarrow \exists$ α-equivariant homeomorphism

$$\phi : \Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$$

If ϕ preserves Patterson–Sullivan measure then α extends to continuous homomorphism

$$\alpha : G_1 \to G_2$$

$G_1 = G_2 = \mathrm{PGL}(2,\mathbb{C})$ simple and connected adjoint real-rank-one Lie group

Γ_i Schottky groups, geometrically finite

Lemma Schottky group $g \geq 2$ Zariski dense in $\mathrm{PGL}(2,\mathbb{C})$

$\hat{\Gamma}$ Zariski closure

(assume connected, else pass to fin index subgroup $\Gamma \cap \hat{\Gamma}_0$ id component with connected closure)

If $\hat{\Gamma}$ connected of dimension $\leq 2 \Rightarrow$ solvable

solvable group cannot contain free group rank $g \geq 2$

then $\dim \hat{\Gamma} = 3 \Rightarrow$ since PGL(2) connected

$$\hat{\Gamma} = \mathrm{PGL}(2)$$

Since F_g no parabolic points $\Rightarrow$ equivariant boundary homeomorphism Φ unique and type-preserving (Tukia)

$\Rightarrow$ $\alpha : \Lambda_1 \to \Lambda_2$ extends to continuous group automorphism $\alpha \in \mathrm{Aut}(\mathrm{PGL}(2, \mathbb{C}))$

$\mathrm{Aut}(\mathrm{PGL}(2, k))$, field k (Schreier and van der Waerden) outer automorphisms from field automorphisms of k

$\Rightarrow \exists$ isomorphism $\Gamma_1 \to \Gamma_2$

$$\gamma_1 \mapsto g \gamma_1^{\sigma} g^{-1}$$

for $g \in \mathrm{PGL}(2, \mathbb{C})$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{R})$

Γ_1 and Γ_2^{σ} conjugate in $\mathrm{PGL}(2, \mathbb{C})$

X_1 and X_2^{σ} isomorphic Riemann surfaces
(X_1 and X_2 conformally or anti-conformally equivalent)