Modular shadows

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Noncommutative geometry and the boundary of modular curves

• Elliptic curves and NC tori (moduli spaces)

• Limiting modular symbols (Manin-M.)

• Levy-Mellin transform (Manin-M.)

• $\mathbb{Q}$-lattices (moduli space and compactification): rational subalgebra (from Connes-M.)

• Classical theory of modular symbols associated with cusp forms as modular pseudomeasures (Manin-M.)

• “Quantum modular forms” (Zagier)

Pseudomeasures (W abelian group)

$$\mu : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to W$$

satisfying: $\forall \alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$,

$$\mu(\alpha, \alpha) = 0,$$

$$\mu(\alpha, \beta) + \mu(\beta, \alpha) = 0,$$

$$\mu(\alpha, \beta) + \mu(\beta, \gamma) + \mu(\gamma, \alpha) = 0.$$

$(\alpha, \beta) \mapsto \mu(\alpha, \beta)$ from Boolean alg of finite unions of positively oriented intervals to W, total mass zero

Abelian group: $(\mu_1 + \mu_2)(\alpha, \beta) := \mu_1(\alpha, \beta) + \mu_2(\alpha, \beta)$
R-module $W \Rightarrow R$-module of pseudomeasures

Universal pseudomeasure

$\nu : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]$ tautological map

$$\mu^U(\alpha,\beta) := \nu(\beta) - \nu(\alpha)$$

values in $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]_0 = \mathrm{Ker}(\epsilon)$

augmentation ideal $\epsilon(\sum_i m_i \nu(\alpha_i)) = \sum_i m_i$

$\forall\ \mu$, $\exists$ homomorphism $w : \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]_0 \to W$

$$\mu = w \circ \mu^U, \quad w(\nu(\beta) - \nu(\alpha)) = \mu(\alpha,\beta)$$

Fractional linear transformations

$\mathrm{GL}_2(\mathbb{Q})$ acting on $\mathbb{P}^1(\mathbb{Q})$

$$(\mu g)(\alpha,\beta) := \mu(g(\alpha), g(\beta))$$

$z \mapsto -z$: even/odd pseudomeasures

Primitive segment:
$I = (g(\infty), g(0))$, for $g \in \mathrm{GL}_2(\mathbb{Z})$
principal homogeneous space over $\mathrm{PSL}_2(\mathbb{Z})$

Infinite: $(-\infty, m)$ and (n, ∞), $m, n \in \mathbb{Z}$.
Finite: (α, β) with $|\alpha - \beta| = n^{-1}$ for some $n \in \mathbb{Z}$, $n \geq 1$

Primitive chain: from α to β
ordered family of primitive segments $I_1, \ldots, I_n$, cyclically matched ends

$g(I_1), \ldots, g(I_n)$ primitive chain from $g(\alpha)$ to $g(\beta)$, for $g \in \mathrm{PSL}_2(\mathbb{Z})$

Convergents

$$I_k(\alpha) := \left(\frac{p_k(\alpha)}{q_k(\alpha)}, \frac{p_{k+1}(\alpha)}{q_{k+1}(\alpha)} \right)$$

$\Rightarrow$ primitive chain from ∞ to α

$\Rightarrow$ Pseudomeasures completely determined by values on primitive segments:

primitive chain $I_1, \ldots, I_n$ from α to $\beta \Rightarrow$

$$\mu(\alpha, \beta) = \sum_{k=1}^{n} \mu(I_k)$$

Premeasure: W-function of primitive segments satisfying $\tilde{\mu}(\alpha, \alpha) = 0$, $\tilde{\mu}(I) + \tilde{\mu}(-I) = 0$, and $\tilde{\mu}(I_1) + \tilde{\mu}(I_2) + \tilde{\mu}(I_3) = 0$ on lenght 3 loop

Uniquely defines pseudo-measure

$$\mu(\alpha, \beta) := \sum_{k=1}^{n} \tilde{\mu}(I_k)$$

Well defined: *elementary moves* on primitive chains

Modular pseudomeasures: $M_W(\Gamma)$

$\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ subgroup of finite index

W left Γ–module $\omega \mapsto g\omega$

$$\mu g(\alpha,\beta) = g\mu(\alpha,\beta)$$

twisted versions: R-valued characters of Γ, W R-module

$$\mu g(\alpha,\beta) = \chi(g)\cdot g\mu(\alpha,\beta)$$

• $\Gamma = \mathrm{SL}_2(\mathbb{Z})$:

$$\mu(\infty,\alpha) = \sum_{k=-1}^{n-1} g_k(\alpha)\mu(\infty,0)$$

$$g_k(\alpha) := \begin{pmatrix} p_k(\alpha) & (-1)^{k+1}p_{k+1}(\alpha) \\ q_k(\alpha) & (-1)^{k+1}q_{k+1}(\alpha) \end{pmatrix} \in SL_2(\mathbb{Z})$$

μ determined by $\mu(\infty,0)$.

• $\{h_k\}$ representatives coset $\Gamma\backslash SL_2(\mathbb{Z})$
$\Rightarrow$ μ determined by $\mu(h_k(\infty),h_k(0))$

A description of $M_W(SL_2(\mathbb{Z}))$

$$M_W(SL_2(\mathbb{Z})) \to W \qquad \mu \mapsto \mu(\infty, 0)$$

injective group homomorphism

Image in $W_+ = \mathrm{Fix}(-id)$ by modularity:

$$(-id)\mu(\infty, 0) = \mu(\infty, 0)$$

Under generators of $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$:

$$(1+\sigma)\mu(\infty, 0) = 0 \qquad (1+\tau+\tau^2)\mu(\infty, 0) = 0$$

Complex

$$0 \to M_W(SL(2,\mathbb{Z})) \to W_+ \to W_+ \times W_+$$

first arrow injective, last arrow $(1+\sigma, 1+\tau+\tau^2)$
$\Rightarrow$ exact: $\mu \mapsto \mu(\infty, 0)$ isomorphism

$M_W(SL(2,\mathbb{Z})) = \mathrm{Ker}(1+\sigma) \cap \mathrm{Ker}(1+\tau+\tau^2)|_{W_+}$

Set $\widetilde{\mu}(g(\infty), g(0)) := g\omega$ for $\omega \in W$
premeasure from $SL_2(\mathbb{Z})$-modular pseudomeasure

$$\mu(g(\alpha), g(\beta)) = \sum_{k=1}^{n} \widetilde{\mu}(g(I_k)) = \sum_{k=1}^{n} g\widetilde{\mu}(I_k) = g\mu(\alpha, \beta)$$

$$\widetilde{\mu}(\alpha, \beta) + \widetilde{\mu}(\beta, \alpha) = g\omega + g\sigma\omega = 0.$$

for $(\beta, \alpha) = g(0, \infty) = g\sigma(\infty, 0)$

Inducing pseudomeasures: reduce $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z})$

Ω totally disconnected compact Hausdorff space

Analytic W–valued pseudo–measure

$$\mu : C(\Omega, W) \to W$$

$C(\Omega, W) = C(\Omega, \mathbb{Z}) \otimes_{\mathbb{Z}} W$ locally constant
W abelian group

• $\mu(V \cup V') = \mu(V) + \mu(V')$ for $V, V' \subset \Omega$ clopen $V \cap V' = \emptyset$.

• $\mu(\Omega) = 0$

(fin.additive function on Boolean alg.gen.by a basis of clopen sets)

Case $\Omega = \partial\mathcal{T}$ boundary of a tree

Currents on a tree $\mathcal{T}$ locally finite tree

$$\mathfrak{c} : \mathcal{T}^{(1)} \to W$$

function on set of oriented edges with

- Momentum conservation at vertices:

$$\sum_{s(e)=v} \mathfrak{c}(e) = 0$$

$s(e)$, $t(e)$ = source and target

- Orientation reversal:

$$\mathfrak{c}(\bar{e}) = -\mathfrak{c}(e)$$

$\bar{e}$ = reverse orientation

Canonical isomorphism: Currents $\Leftrightarrow$ Analytic pseudomeasures

$$\mu(V(e)) = \mathfrak{c}(e)$$

The tree of $\mathsf{PSL}_2(\mathbb{Z})$ (embedded in $\mathbb{H}$)

Vertices: elliptic points $\widetilde{I} \cup \widetilde{R}$
$\widetilde{I} = \mathsf{PSL}(2,\mathbb{Z}) \cdot i$
$\widetilde{R} = \mathsf{PSL}(2,\mathbb{Z}) \cdot e^{2\pi i/3}$

Edges: geodesic arcs
$\{\gamma(i), \gamma(e^{2\pi i/3})\}$, for $\gamma \in \mathsf{PSL}(2,\mathbb{Z})$

Boundary $\partial\mathcal{T}$:

$\exists$ continuous $PSL(2,\mathbb{Z})$–equivariant surjection

$$\Upsilon : \partial\mathcal{T} \to \partial\mathbb{H} = \mathbb{P}^1(\mathbb{R})$$

one–to–one on $\mathbb{P}^1(\mathbb{R}) \cap (\mathbb{R} \smallsetminus \mathbb{Q})$
two–to–one on $\mathbb{P}^1(\mathbb{Q})$

Proof: Farey tessellation of hyperbolic plane $\mathbb{H}$

Disconnection space

For $U \subset \mathbb{P}^1(\mathbb{R})$ abelian C^*-algebra $\mathcal{A}_U$ gen. by $C(\mathbb{P}^1(\mathbb{R}))$ and characteristic functions of intervals w/ endpoints in U.

$$\mathcal{A}_U = C(D_U)$$

U dense $\Rightarrow$ D_U totally disconnected

$\Upsilon : \partial\mathcal{T} \to \mathbb{P}^1(\mathbb{R})$ factors: homeomorphism

$$\widetilde{\Upsilon} : \partial\mathcal{T} \to D_{\mathbb{P}^1(\mathbb{Q})}$$

followed by surjection

$$D_{\mathbb{P}^1(\mathbb{Q})} \to \mathbb{P}^1(\mathbb{R}) \Leftrightarrow C(\mathbb{P}^1(\mathbb{R})) \hookrightarrow C(D_{\mathbb{P}^1(\mathbb{Q})})$$

Pseudomeasures and currents

Natural bijection:

W-pseudomeas. on $\mathbb{P}^1(\mathbb{R}) \Leftrightarrow$ analytic on $D_{\mathbb{P}^1(\mathbb{Q})}$

$$\mu_{an}(V(e)) := \mu(\alpha, \beta)$$

$\mu(\beta,\alpha) = -\mu(\alpha,\beta) \Rightarrow \mu_{an}(\partial \mathcal{T}) = 0$

$\mu(\alpha,\beta)+\mu(\beta,\gamma)+\mu(\gamma,\alpha) = 0 \Rightarrow \mu_{an}(V \cup V') = \mu_{an}(V) + \mu_{an}(V')$ for $V \cap V' = \emptyset$

$$\mu(\alpha, \beta) := \mu_{an}(V(e)) \quad \text{for} \quad \Upsilon V(e) = [\alpha, \beta]$$

$\mu_{an}(V \cup V') = \mu_{an}(V) + \mu_{an}(V')$ for $V \cap V' = \emptyset$ and $\mu_{an}(\partial \mathcal{T}) = 0 \Rightarrow \mu(\beta,\alpha) = -\mu(\alpha,\beta)$ and $\mu(\alpha,\beta)+\mu(\beta,\gamma)+\mu(\gamma,\alpha) = 0$

$\Gamma-$modular $W-$valued

Cohomology $\mu \in M_W(\Gamma)$ and $\alpha \in \mathbb{P}^1(\mathbb{Q})$

$$c_\alpha^\mu = c_\alpha : \Gamma \to W \qquad c_\alpha(g) := \mu(g\alpha, \alpha)$$

1–cocycle $Z^1(\Gamma, W)$

$$c_\alpha(gh) = c_\alpha(g) + g c_\alpha(h)\,.$$

changing α cohomologous

$$c_\alpha(g) - c_\beta(g) = g\mu(\alpha, \beta) - \mu(\alpha, \beta)$$

Γ_α fixing $\alpha \Rightarrow c_\alpha|_{\Gamma_\alpha} = 0$
cuspidal $c \in H^1(\Gamma, W)$ if all $c|_{\Gamma_\alpha} = 0$

$$M_W(\Gamma) \to H^1(\Gamma, W)_{cusp}$$

Bijection

$$M_W(\mathsf{PSL}_2(\mathbb{Z})) \cong Z^1(\mathsf{PSL}_2(\mathbb{Z}), W_+)_{cusp}$$

using bijection

$$Z^1(\mathsf{PSL}_2(\mathbb{Z}), W_+) \to \mathsf{Ker}(1+\sigma)\times\mathsf{Ker}(1+\tau+\tau^2)$$

in $W_+ \times W_+$ given by

$$c \mapsto (c(\sigma), c(\tau))$$

and $\mathsf{PSL}_2(\mathbb{Z})$-modular pseudomeasures have

$$c^\mu_\infty(\sigma) = c^\mu_\infty(\tau) = -\mu(\infty, 0) \in W_+$$

Generalized Dedekind symbols: (Fukuhara)
$V := \{(p,q) \in \mathbb{Z}^2 \,|\, p \geq 1, \mathrm{gcd}(p,q) = 1\}$
W–valued generalized Dedekind symbol:
$D : V \to W$ with functional equation

$$D(p,q) = D(p, q+p)$$

Associated *reciprocity function*:

$$R(p,q) := D(p,q) - D(q,-p)$$

on $V_0 := \{(p,q) \in V \,|\, q \geq 1\}$, satisfying functional equation

$$R(p+q,q) + R(p,p+q) = R(p,q)$$

$R(1,1) = 0$ if D even under $(p,q) \mapsto (p,-q)$

D determined by R (when no 2-torsion)

Reciprocity functions

Π = set of primitive segments in $[0,1]$

$$\Pi \to V_0 \qquad I = [a/p, b/q] \mapsto (p,q)$$

bijection

W-valued pseudomeasure μ, for $n \in \mathbb{Z}$:

$$R_{\mu,n}(p,q) := \mu\left(n + \frac{a}{p}, n + \frac{b}{q}\right)$$

$\Rightarrow$ reciprocity functions with $R_{\mu,n}(1,1) = 0$ iff $\mu(n, n+1) = 0$

To check functional equation: for $n = 0$, properties of pseudomeasure

$$\mu(\alpha,\beta) + \mu(\beta,\gamma) + \mu(\gamma,\alpha) = 0$$

applied to the Farey triple

$$\alpha = \frac{a}{p},\ \beta = \frac{a+b}{p+q},\ \gamma = \frac{b}{q}$$

Then shift this triple by n

Also have $R_{\mu,0}(1,1) = \mu(0,1)$

The other way:

Given sequence of reciprocity functions R_n, $n \in \mathbb{Z}$, and a $\omega \in W$

$$\widetilde{\mu}(-\infty, 0) := \omega$$

$$\widetilde{\mu}(-\infty, n) := \omega + R_0(1,1) + R_1(1,1) + \cdots + R_{n-1}(1,1)$$

$$\widetilde{\mu}(-\infty, -n) := \omega - R_{-1}(1,1) - \cdots - R_{-n}(1,1)$$

$$\widetilde{\mu}(-n, \infty) := R_{-n}(1,1) + R_{-n+1}(1,1) + \cdots + R_{-1}(1,1) - \omega$$

$$\widetilde{\mu}(n, \infty) := -R_{n-1}(1,1) - R_{n-2}(1,1) - \cdots - R_0(1,1) - \omega$$

Positively oriented finite segments:

$$\widetilde{\mu}(n+\alpha, n+\beta) := R_n(p,q)$$

if $0 \leq \alpha = a/p < \beta = p/q \leq 1$

Negatively oriented primitive segments:

$$\widetilde{\mu}(\beta, \alpha) = -\widetilde{\mu}(\alpha, \beta)$$

$\Rightarrow$ $\widetilde{\mu}$ premeasure $\Rightarrow$ pseudomeasure

Holomorphic functions vanishing at cusps:

$\mathcal{O}(\mathbb{H})_{cusp}$: holomorphic functions $f(z)$ along geodesics to cusp $\alpha \in \mathbb{P}^1(\mathbb{Q})$ vanish faster than $\exp(\ell_{(z_0,z)})^{-N}$, all N

$$(\alpha, \beta) \mapsto \int_\alpha^\beta f(z)dz$$

integration along geodesic

Element in linear dual $\mathbb{W} := (\mathcal{O}(\mathbb{H})_{cusp})^*$
$\Rightarrow$ $\mathbb{W}$-valued pseudomeasure μ

Action of Hecke operators on pseudomeasures If W has left $\mathrm{GL}_2^+(\mathbb{Q})$ action

$GL_2^+(\mathbb{Q})$-bimodule $M_W(\Gamma)$ with right action

$$(\mu g)(\alpha,\beta) := \mu(g(\alpha), g(\beta))$$

and left action

$$(g\mu)(\alpha,\beta) := g(\mu(\alpha,\beta))$$

Δ finite union of double cosets in $\Gamma\backslash\mathrm{GL}_2^+(\mathbb{Q})/\Gamma$ with $\{\delta_i\}$ complete finite family of representatives

$\Rightarrow T_\Delta : M_W(\Gamma) \to M_W(\Gamma)$

$$T_\Delta : \mu \mapsto \mu_\Delta := \sum_i \delta_i^{-1}\mu\delta_i$$

depends only on Δ

Classical Hecke operators T_n for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$:
Δ_n double coset classes from integer matrices with det $= n$ have complete system of representatives

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad ad = n,\ 1 \le b \le d$$

$T_n := T_{\Delta_n}$

Pseudomeasures from cusp forms

Right action of weight $w+2$ on holomorphic (or meromorphic) functions on $\mathbb{H}$:

$$f|[g]_{w+2}(z) := (\det g)^{w+1} f(gz) j(g,z)^{-(w+2)}$$

with $j(g,z) := cz + d$ and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Right action of $GL_2(\mathbb{R})$ on polynomials in two variables: $(Pg)(X,Y)$ given by

$$P((\det g)^{-1}(aX + bY), (\det g)^{-1}(cX + dY))$$

$$\int_{g\alpha}^{g\beta} f(z)P(z,1)dz = \int_{\alpha}^{\beta} f(gz)P(gz,1)d(gz) =$$

$$\int_{\alpha}^{\beta} f|[g]_{w+2}(z)P((\det g)^{-1}(az+b),(\det g)^{-1}(cz+d))dz =$$

$$\int_{\alpha}^{\beta} f|[g]_{w+2}(z)(Pg)(z,1)dz$$

If for a finite family of $g_k \in GL_2^+(\mathbb{Q})$ and $c_k \in \mathbb{C}$ have

$$\sum_k c_k f|[g_k]_{w+2}(z) = \lambda f(z)$$

then

$$\sum_k c_k \int_{g_k\alpha}^{g_k\beta} f(z)P(z,1)dz =$$

$$\int_{\alpha}^{\beta} f(z)\lambda \sum_k c_k(Pg_k)(z,1)dz$$

Shadows of modular symbols

$\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ finite index, $w \in \mathbb{N}$
cusp form $f(z)$ of weight $w+2$ for Γ:
holomorphic on $\mathbb{H}$, vanishing at cusps with

$$\sum_k c_k f|[g_k]_{w+2}(z) = f(z)$$

($\lambda = 1$ or $\lambda = \chi(g)$ with character)

$S_{w+2}(\Gamma) = \mathbb{C}$-vector space of cusp forms
$F_w =$ space of polynomials of degree w in two variables
$\mathbb{W} =$ linear functionals on $S_{w+2}(\Gamma) \otimes F_w$

$$\mu(\alpha, \beta) : f \otimes P \mapsto \int_\alpha^\beta f(z) P(z, 1) dz$$

$\mathbb{W}$-valued pseudomeasure *shadow* of the modular symbol

Lévy functions and Lévy–Mellin transform

Classical Lévy function

$$L(f)(\alpha) := \sum_{n=0}^{\infty} f(q_n(\alpha), q_{n+1}(\alpha))$$

$q_n(\alpha)$, $n \geq 0$, denominators of convergents of $\alpha \in \mathbb{R}$

f function of pairs $(q', q) \in \mathbb{Z}^2$, $1 \leq q' \leq q$, gcd$(q', q) = 1$, valued in top group, rapidly decreasing

$L(f)(\alpha)$ continuous on irrationals and period 1

For $(q', q) \neq (1, 1)$ as above all $\alpha \in [0, 1/2]$ with $(q', q) = (q_n(\alpha), q_{n+1}(\alpha))$ fill primitive I length $(q(q + q'))^{-1}$ and symmetric $1 - I$

$\Rightarrow$ (formal) infinite linear combinations of characteristic functions of primitive segments I

$$L(f) := \sum_I f(I)\chi_I$$

$I \mapsto f(I)$ Lévy function

Formal Dirichlet series:

$A = \{a_1, \dots, a_n, \dots\}$ sequence of elements of abelian gr $\mathcal{A}$

$$L_A(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Composition

$$(a_1, \dots, a_n, \dots) \cdot (b_1, \dots, b_n, \dots) = (c_1, \dots, c_n, \dots)$$

$$c_n := \sum_{d_1, d_2 \colon d_1 d_2 = n} a_{d_1} \cdot b_{d_2}$$

$$L_C(s) = L_A(s) \cdot L_B(s).$$

Set L: pairs (c,d), $1 \le c < d \in \mathbb{Z}$, $\gcd(c,d)=1$
Set R: pairs of reduced matrices (g^-, g^+) same lower row, $\det g^\pm = \pm 1$
Set S: pairs of primitive segments (I^-, I^+), length $<$ $1/2$, $I^- \subset [0, \frac{1}{2}]$, $I^+ = 1 - I^- \subset [\frac{1}{2}, 1]$. Map $R \to S$:

$$(g^-_{c,d}, g^+_{c,d}) \mapsto \left(I^-_{c,d} := [g^-_{c,d}(0), g^-_{c,d}(1)], I^+_{c,d} := [g^+_{c,d}(0), g^+_{c,d}(1)] \right)$$

W left $GL_2^+(\mathbb{Q})$-module and $\mu \in M_W(\mathrm{SL}_2(\mathbb{Z}))$
Lévy function

$$f_\mu(I^-_{c,d})(s) := \frac{1}{|I^-_{c,d}| d^s} \begin{pmatrix} 1 & -cd^{-1} \\ 0 & d^{-1} \end{pmatrix} \mu(\infty, \frac{c}{d})$$

supported on prim segments in $[0, 1/2]$

values in grp formal Dirichlet series w/ coefficients in W one non-vanishing term

Lévy–Mellin transform (shadows of classical Mellin transforms)

Def: Lévy–Mellin transform

$$LM_\mu(s) := \int_0^{1/2} L(f_\mu)(\alpha, s) d\alpha$$

Formal Dirichlet series w/ coeffs in $\mathbb{Z}[\mathrm{GL}_2^+(\mathbb{Z})]$:

$$Z_-(s) := \sum_{d_1=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & d_1^{-1} \end{pmatrix} \frac{1}{d_1^s}$$

$$Z_+(s) := \sum_{d_2=1}^{\infty} \begin{pmatrix} d_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{d_2^s}$$

Lévy–Mellin transform and Hecke operators:

$$Z_+(s) \cdot Z_-(s) \cdot LM_\mu(s) = \sum_{n=1}^{\infty} \frac{(T_n\mu)(\infty, 0)}{n^s}$$

$T_n =$ Hecke operator

Observations:

• Lévy–Mellin applied to pseudomeasures from cusp form $\Rightarrow$ usual Mellin transform: integral on boundary instead of imaginary line

• Generalization: iterated Lévy–Mellin transforms for Manin's iterated Shimura integrals (iterated integrals)

Pseudomeasures and boundary geometry (K-theory)

$\mathcal{A} = C(\partial\mathcal{T} \times \mathbb{P})$ with action $G = \mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

$$K_0(\mathcal{A} \rtimes G) = Coker(\alpha), \quad K_1(\mathcal{A} \rtimes G) = Ker(\alpha),$$

$$\alpha : C(\partial\mathcal{T}\times\mathbb{P},\mathbb{Z}) \to C(\partial\mathcal{T}\times\mathbb{P},\mathbb{Z})^{G_2} \oplus C(\partial\mathcal{T}\times\mathbb{P},\mathbb{Z})^{G_3}$$

$$\alpha : f \mapsto (f + f\circ\sigma, f + f\circ\tau + f\circ\tau^2)$$

σ and τ generators of $G_2 = \mathbb{Z}/2\mathbb{Z}$ and $G_3 = \mathbb{Z}/3\mathbb{Z}$

Proof: PV exact sequence, $\partial\mathcal{T}$ totally disconnected

Pseudomeasures and integration

Analytic pseudomeasure:

$$f \mapsto \int f \, d\mu \ \in W \qquad f \in C(D_{\mathbb{P}^1(\mathbb{Q})}, \mathbb{Z})$$

$f = \sum_{i=1}^{n} a_i \chi_{I_i}$ with $\Upsilon I_i = g_i(\infty, 0) \subset \mathbb{P}^1(\mathbb{R})$

$$\int f \, d\mu = \sum_i a_i \mu(I_i) \in W.$$

Change of variable formula

$$\int f \circ g \, d\mu = \int f \, d(\mu \circ g^{-1}).$$

- G-modular $W-$valued pseudomeasure μ

$h \in K_1(\mathcal{A} \rtimes G) \Rightarrow$

$$\mu_h(\infty, 0) := \int h \, d\mu$$

G-modular $W-$valued pseudomeasure

Proof:

$\int h \, d\mu$ annihilated by $1 + \sigma$ and $1 + \tau + \tau^2$:

$$K_1(\mathcal{A} \rtimes G) = \mathrm{Ker}(\alpha)$$

$h + h \circ \sigma = 0$ and $h + h \circ \tau + h \circ \tau^2 = 0$

• G-modular W–valued pseudo–measure μ
$\Rightarrow$ group homomorphism $\mu : K_0(\mathcal{A} \rtimes G) \to W$

Proof: $\mu : C(\partial\mathcal{T}, \mathbb{Z}) \to W$

$$mu(f) = \int f \, d\mu$$

descends to quotient of $C(\partial\mathcal{T}, \mathbb{Z})$ by $f + f \circ \sigma$ and $f + f \circ \tau + f \circ \tau^2$:

$$\int (f + f \circ \sigma) d\mu = \int f \, d\mu + \int f \, d\mu \circ \sigma = (1 + \sigma) \int f \, d\mu$$

$$\int \chi_{(g(\infty), g(0))} \, d\mu = \int \chi_{(\infty,0)} \circ g^{-1} \, d\mu = \int \chi_{(\infty,0)} \, d\mu \circ g$$

$$\sigma \int \chi_{(\infty,0)} \, d\mu \circ g = \int \chi_{(\infty,0)} d\mu \circ g \circ \sigma = \int \chi_{(g\sigma(\infty), g\sigma(0))} \, d\mu,$$

$$(1 + \sigma) \int \chi_{(g(\infty), g(0))} \, d\mu = g(1 + \sigma)\mu(\infty, 0) = 0.$$

similar for $f + f \circ \tau + f \circ \tau^2$

Shift space and boundary NCG

Generalized Gauss shift:

$$\widehat{T} : x \mapsto -\mathsf{sign}(x) T(|x|),$$

with

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right].$$

finite index subgroup $\Gamma \subset G = \mathrm{PSL}_2(\mathbb{Z})$

$$\widehat{T}_{\mathbb{P}} : \mathcal{I} \times \mathbb{P} \to \mathcal{I} \times \mathbb{P}, \quad (x, s) \mapsto (\hat{T}(x), [gST^k]),$$

with $\mathcal{I} = [-1, 1] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ and $k = sign(x) n_1$
n–th iterate of $\widehat{T}_{\mathbb{P}}$ acts on $\mathbb{P}$

$$\begin{pmatrix} -\mathsf{sign}(x_1) p_{k-1}(x) & (-1)^k p_k(x) \\ q_{k-1}(x) & (-1)^{k+1} q_k(x) \end{pmatrix}.$$

(C.H. Chang – D. Mayer, Manin – M., Kessenböhmer – Stratmann)

Countable alphabet $\mathfrak{A} = \mathbb{Z}^\times \times \mathbb{P}$ and *admissible sequences*

$$\Sigma_\Gamma = \{((x_1, s_1), (x_2, s_2), \ldots) \mid A_{(x_i,s_i),(x_{i+1},s_{i+1})} = 1\},$$

with $A_{(x,s),(x',s')} = 1$ if $xx' < 0$ and

$$s' = \tau_x(s) := [gST^{x_1}] \in \mathbb{P}, \quad \text{where } s = \Gamma g$$

and $A_{(x,s),(x',s')} = 0$ otherwise

action of $\widehat{T}_{\mathbb{P}}$ one–sided shift $\sigma : \Sigma_\Gamma \to \Sigma_\Gamma$

$$\sigma : ((x_1, s_1), (x_2, s_2), \ldots) \mapsto ((x_2, s_2), (x_3, s_3), \ldots)$$

NCG of boundary of modular curves:

• $C(\mathbb{P}^1(\mathbb{R})) \rtimes G$

• $C(D_{\mathbb{P}^1(\mathbb{Q})}) \rtimes G$

• Exel-Laca algebra: countably infinite alphabet $\mathfrak{A}$ and admissibility matrix A

- $S_a^* S_a$ and $S_b^* S_b$ commute for all $a, b \in \mathfrak{A}$
- $S_a^* S_b = 0$ for $a \neq b$
- $(S_a^* S_a) S_b = A_{ab} S_b$ for all $a, b \in \mathfrak{A}$
- For any pair of finite subsets $X, Y \subset \mathfrak{A}$ with

$$A(X, Y, b) := \prod_{x \in X} A_{xb} \prod_{y \in Y} (1 - A_{yb}) = 0$$

for all but fin many $b \in \mathfrak{A} \Rightarrow$

$$\prod_{x \in X} S_x^* S_x \prod_{y \in Y} (1 - S_y^* S_y) = \sum_{b \in \mathfrak{A}} A(X, Y, b) S_b S_b^*$$

$\mathcal{J} = \Upsilon^{-1}[-1,1] \subset D_{\mathbb{P}^1(\mathbb{Q})}$, $\mathcal{J}_{+1} = \Upsilon^{-1}[0,1]$, $\mathcal{J}_{-1} = \Upsilon^{-1}[-1,0]$, and

$$\mathcal{J}_k = \{x \in \mathcal{J} \mid \Upsilon(x) = sign(k)[a_1, a_2, \ldots], \, a_1 = |k|, \, a_i \geq 1\}$$

$$\mathcal{J}_{k,s} := \mathcal{J}_k \times \{s\},$$

for $s \in \mathbb{P} = \Gamma \backslash G$ and $k \in \mathbb{Z}^\times$, and $\chi_{k,s}$ = char function of $\mathcal{J}_{k,s}$

Subalgebra of $C(D_{\mathbb{P}^1(\mathbb{Q})} \times \mathbb{P}) \rtimes G$ gen by

$$S_{k,s} := \chi_{k,s} U_k,$$

with $U_k = U_{\gamma_k}$ and $\gamma_k = T^k S \in \Gamma$

isomorphic to Exel–Laca algebra O_A of the shift (Σ_Γ, σ)

Limiting modular pseudomeasures

Lyapunov exponent of the Gauss shift:

$$\lambda(\theta) := \lim_{n\to\infty} \frac{2\log q_n(\theta)}{n}$$

defined away from an exceptional set $\Omega \subset \mathbb{P}^1(\mathbb{R})$

Limiting modular symbol

$$\{\{*,\theta\}\} = \lim_{t\to\infty} \frac{1}{t}\{x_0, y(t)\}_\Gamma$$

computed by the limit

$$\lim_{n\to\infty} \frac{1}{\lambda(\theta)n} \sum_{k=1}^{n+1} \{g_{k-1}(0), g_{k-1}(\infty)\}_\Gamma \in H_1(X_\Gamma, \mathbb{R})$$

with $g_k = g_k(\theta)$ action of k-th power of Gauss shift

$W-$valued pseudomeasure μ
(analytic pseudom. on $D_{\mathbb{P}^1(\mathbb{Q})}$)

Define $\mu^{lim}(\infty,\theta)$ as

$$\lim_{n\to\infty}\frac{1}{\lambda(\theta)n}\sum_{k=1}^{n+1}\mu(g_{k-1}(\infty),g_{k-1}(0))\in W$$

away from an exc. set $\Omega\supset\mathbb{P}^1(\mathbb{Q})$

$$\mu^{lim}(\theta,\infty)=\lim_{n\to\infty}\frac{1}{\lambda(\theta)n}\sum_{k=1}^{n+1}\mu(g_{k-1}(0,\infty))$$

$$\mu^{lim}(\theta,\eta):=\mu^{lim}(\theta,\infty)+\mu^{lim}(\infty,\eta)$$

satisfying

$$\mu^{lim}(\eta,\theta)=-\mu^{lim}(\theta,\eta)$$

$$\mu^{lim}(\theta,\eta)+\mu^{lim}(\theta,\zeta)+\mu^{lim}(\zeta,\theta)=0$$

μ^{lim} is $G-$modular if μ is $G-$modular

Current

$$\mu^{lim}(\infty,\theta)=\lim_{n\to\infty}\frac{1}{\lambda(\theta)n}\sum_{k=1}^{n}\mathbf{c}(e_k)$$