Variants of Equivariant Seiberg-Witten Floer Homology

Matilde Marcolli, Bai-Ling Wang

Abstract

For a rational homology 3-sphere $Y$ with a Spin$^c$ structure $s$, we show that simple algebraic manipulations of our construction of equivariant Seiberg-Witten Floer homology in [5] lead to a collection of variants $HF_{*, U(1)}^{SW, -}(Y, s), HF_{*, U(1)}^{SW, \infty}(Y, s), HF_{*, U(1)}^{SW, +}(Y, s), \hat{HF}_{*}^{SW}(Y, s)$ and $HF_{*, U(1)}^{SW, \text{red}, s}(Y, s)$ which are topological invariants. We establish a long exact sequence relating $HF_{*, U(1)}^{SW, \pm}(Y, s)$ and $HF_{*, U(1)}^{SW, \infty}(Y, s)$. We show they satisfy a duality under orientation reversal, and we explain their relation to the equivariant Seiberg-Witten Floer (co)homologies introduced in [5]. We conjecture the equivalence of these versions of equivariant Seiberg-Witten Floer homology with the Heegaard Floer invariants introduced by Ozsváth and Szabó.

Key words: rational homology 3-spheres, equivariant Seiberg-Witten Floer homology, Spin$^c$ structures, topological invariants.


1 Introduction

For any rational homology 3-sphere $Y$ with a Spin$^c$ structure $s$, we constructed in [5] an equivariant Seiberg-Witten Floer homology $HF_{*, U(1)}^{SW}(Y, s)$, which is a topological invariant. In this paper, we will generalize this construction to provide a collection of equivariant Seiberg-Witten Floer homologies $HF_{*, U(1)}^{SW, -}(Y, s), HF_{*, U(1)}^{SW, \infty}(Y, s), HF_{*, U(1)}^{SW, +}(Y, s), \hat{HF}_{*}^{SW}(Y, s)$ and $HF_{*, U(1)}^{SW, \text{red}, s}(Y, s)$, all of which are topological invariants, such that $HF_{*, U(1)}^{SW, +}(Y, s)$ is isomorphic to the equivariant Seiberg-Witten Floer homology $HF_{*, U(1)}^{SW}(Y, s)$ constructed in [5]. The construction utilizes the $U(1)$-invariant forms on $U(1)$-manifolds twisted with coefficients in the Laurent polynomial algebra over integers.

In analogy to Austin and Broom’s construction of equivariant instanton Floer homology in [1], the equivariant Seiberg-Witten Floer homology $HF_{*, U(1)}^{SW}(Y, s)$ is the homology of the complex
\( (CF^\text{SW}_{s,U(1)}(Y,s), D) \), where \( CF^\text{SW}_{s,U(1)}(Y,s) \) is generated by equivariant de Rham forms over all \( U(1) \)-orbits of the solutions of 3-dimensional Seiberg-Witten equations on \( (Y,s) \) modulo based gauge transformations (Cf.\[5\]). More specifically,

\[
CF^\text{SW}_{s,U(1)}(Y,s) = \bigoplus_{a \in \mathcal{M}_y^s(s)} \mathbb{Z}[\Omega] \otimes (\mathbb{Z}\eta_a \oplus \mathbb{Z}1_a) \oplus \mathbb{Z}[\Omega] \otimes \mathbb{Z}1_\theta, \tag{1}
\]

where \( \mathcal{M}_y(s) = \mathcal{M}_y^s(s) \cup \{\theta\} \) is the equivalence classes of solutions to the Seiberg-Witten equations for a good pair of metric and perturbations, consists of the irreducible monopoles \( \mathcal{M}_y^s(s) \) and the unique reducible monopole \( \theta \). We used the notation \( \eta_a \) to denote a 1-form on \( O_a \cong S^1 \), such that the cohomology class \([\eta_a]\) is an integral generator of \( H^1(O_a) \). Similarly, we denote by \( 1_a \) the 0-form given by the constant function.

Each generator is endowed with a grading such that, for any \( k \geq 0 \),

\[
gr(\Omega^k \otimes \eta_a) = 2k + gr(a), \quad gr(\Omega^k \otimes 1_a) = 2k + gr(a) + 1, \quad \text{and} \quad gr(\Omega^k \otimes 1_\theta) = 2k, \tag{2}
\]

where \( gr : \mathcal{M}_y^s(s) \to \mathbb{Z} \) is the relative grading with respect to the reducible monopole \( \theta \). This corresponds to grading equivariant de Rham forms on each orbit \( O_a \) by codimension (Cf.\[5\] §5 for details).

The differential operator \( D \) can be expressed explicitly in components as the form:

\[
D(\Omega^k \otimes \eta_a) = \sum_{b \in \mathcal{M}_y^s(s)} n_{ab} \Omega^k \otimes \eta_b + \sum_{c \in \mathcal{M}_y^s(s)} m_{ac} \Omega^k \otimes 1_c - \Omega^{k-1} \otimes 1_a
\]

\[
+ n_{a\theta} \Omega^k \otimes 1_\theta \text{ (if } gr(a) = 1); \tag{3}
\]

\[
D(\Omega^k \otimes 1_a) = - \sum_{b \in \mathcal{M}_y^s(s)} n_{ab} \Omega^k \otimes 1_b;
\]

\[
D(\Omega^k \otimes 1_\theta) = \sum_{d \in \mathcal{M}_y^s(s)} n_{\theta d} \Omega^k \otimes 1_d.
\]

where \( n_{ab}, n_{a\theta}, \text{ and } n_{\theta d} \) is the counting of flowlines from \( a \) to \( b \) (if \( gr(a) - gr(b) = 1 \)), from \( a \) to \( \theta \) (if \( gr(a) = 1 \)) and from \( \theta \) to \( d \) (if \( gr(d) = -2 \)), and \( m_{ac} \) (if \( gr(a) - gr(c) = 2 \)) is described as a relative Euler number associated to the 2-dimensional moduli space of flowlines from \( a \) to \( c \) (Cf. Lemma 5.7 of \[5\]). In the next section, we shall briefly review the construction and various relations among the coefficients, as established in \[5\]. These identities ensure that \( D^2 = 0 \). Notice that, in the complex \( CF^\text{SW}_{s,U(1)}(Y,s) \) and in the expression of the differential operator, only terms with non-negative powers of \( \Omega \) are considered. We modify the construction as follows.
Definition 1.1. Let $\text{CF}_{s,U[1]}^{SW,\infty}(Y, s)$ be the graded complex generated by

$$\{\Omega^k \otimes \eta, \Omega^k \otimes 1_a, \Omega^k \otimes 1_\theta : a \in \mathcal{M}^\ast_Y(s), k \in \mathbb{Z}\}$$

with the grading $\text{gr}$ and the differential operator $D$ given by (2) and (3) respectively. Let $\text{CF}_{s,U[1]}^{SW,-}(Y, s)$ be the subcomplex of $\text{CF}_{s,U[1]}^{SW,\infty}(Y, s)$, generated by those generators with negative power of $\Omega$. The quotient complex is denoted by $\text{CF}_{s,U[1]}^{SW,+}(Y, s)$. Their homologies are denoted by $HF_{s,U[1]}^{SW,\infty}(Y, s)$, $HF_{s,U[1]}^{SW,-}(Y, s)$ and $HF_{s,U[1]}^{SW,+}(Y, s)$ respectively.

The main results in this paper relate these homologies to the equivariant Seiberg-Witten-Floer homology $HF_{s,U[1]}^{SW}(Y, s)$ and cohomology $HF_{U[1]}^{SW}(Y, s)$ constructed in [5] and establish some of their main properties.

Theorem 1.2. For any rational homology 3-sphere $Y$ with a Spin$^c$ structure $s \in \text{Spin}^c(Y)$, these homologies satisfy the following properties:

1. $HF_{s,U[1]}^{SW,\infty}(Y, s) \cong \mathbb{Z}[\Omega, \Omega^{-1}]$.

2. $HF_{s,U[1]}^{SW,+}(Y, s) \cong HF_{s,U[1]}^{SW,-}(Y, s)$ where $HF_{s,U[1]}^{SW}(Y, s)$ is the equivariant Seiberg-Witten Floer homology for $(Y, s)$ constructed in [5].

3. $HF_{s,U[1]}^{SW,-}(Y, s) \cong HF_{U[1]}^{SW,-}(Y, s)$ where $HF_{U[1]}^{SW,-}(Y, s)$ is the equivariant Seiberg-Witten Floer cohomology for $(Y, s)$ constructed in [5].

4. There exists a long exact sequence

$$\cdots \rightarrow HF_{s,U[1]}^{SW,-}(Y, s) \xrightarrow{\iota} HF_{s,U[1]}^{SW,\infty}(Y, s) \xrightarrow{\pi} HF_{s,U[1]}^{SW,+}(Y, s) \xrightarrow{\delta} HF_{s-1,U[1]}^{SW,-}(Y, s) \rightarrow \cdots \quad (4)$$

relating these homologies. Moreover, $HF_{s,U[1]}^{SW,-}(Y, s)$, $HF_{s,U[1]}^{SW,\infty}(Y, s)$, $HF_{s,U[1]}^{SW,+}(Y, s)$ and $HF_{\text{red},s}^{SW}(Y, s) = \text{Coker}(\pi) \cong \text{Ker}(\delta_{s-1})$ are all topological invariants of $(Y, s)$.

5. There is a $u$-action on $HF_{s,U[1]}^{SW,-}(Y, s)$, $HF_{s,U[1]}^{SW,\infty}(Y, s)$ and $HF_{s,U[1]}^{SW,+}(Y, s)$ respectively which decreases the degree by two, and is related to the cutting down moduli spaces of flowlines by a geometric representative of a degree 2 characteristic form. The long exact sequence (4) is a long exact sequence of $\mathbb{Z}[u]$-modules.
6. There is a homology group $\widehat{HF}_s^{SW}(Y,s)$, which is also a topological invariant of $(Y,s)$, such that the following sequence is exact:

$$\cdots \to \widehat{HF}_s^{SW}(Y,s) \xrightarrow{\cdot u} H_{{s,U[1]}\{+\}}^{SW}(Y,s) \xrightarrow{\cdot \Omega_{-1}} H_{{s,U[1]}\{-\}}^{SW}(Y,s) \to \widehat{HF}_{s-1}^{SW}(Y,s) \to \cdots \tag{5}$$

and that $\widehat{HF}_s^{SW}(Y,s)$ is non-trivial if and only if $H_{{s,U[1]}\{+\}}^{SW}(Y,s)$ is non-trivial.

The $u$-action in the main theorem is induced from a $u$-action on the chain complex

$$u : \quad CF_{s,U[1]}^{SW,\infty} \to CF_{s,U[1]}^{SW,\infty},$$

which decreases the degree by 2. We will show that this $u$-action is homotopic to the obvious $\Omega_{-1}$ action on the chain complex $CF_{s,U[1]}^{SW,\infty}$. Thus, the induced $u$-action on $H_{{s,U[1]}\{+\}}^{SW}(Y,s)$ endows them with $\mathbb{Z}[u]$-module structures.

Let $\overline{CF}_s^{SW}(Y,s)$ be the subcomplex of $CF_{s,U[1]}^{SW,\{+\}}(Y,s)$ such that the following sequence is a short exact sequence of chain complexes:

$$0 \to \overline{CF}_s^{SW}(Y,s) \to CF_{s,U[1]}^{SW,\{+\}}(Y,s) \xrightarrow{\Omega_{-1}} CF_{s,U[1]}^{SW,\{+\}}(Y,s) \to 0$$

We can define $\widehat{HF}_s^{SW}(Y,s)$ to be the homology of $\overline{CF}_s^{SW}(Y,s)$.

In recent work [7] [8], Ozsváth and Szabó introduced Heegaard Floer invariants $HF_s^{\{\pm\}}(Y,s)$, $HF_s^{\infty}(Y,s)$, $\widehat{HF}_s(Y,s)$, and $HF_{\text{red},s}(Y,s)$, with exact sequences relating them. In view of their construction, the result of Theorem 1.2, together with the identification of our $H_{{s,U[1]}\{\pm\}}^{SW}(Y,s)$ and the $HF_s^{\infty}(Y,s)$ of Ozsváth and Szabó, suggest the following conjecture.

**Conjecture 1.3.** For any rational homology 3-sphere $Y$ with a Spin$^c$ structure $s \in \text{Spin}^c(Y)$, there are isomorphisms

$$HF_{s,U[1]}^{SW,\{\pm\}}(Y,s) \cong HF_{s}^{\{\pm\}}(Y,s), \quad \widehat{HF}_s^{SW}(Y,s) \cong \widehat{HF}_s(Y,s), \quad HF_{\text{red},s}^{SW}(Y,s) \cong HF_{\text{red},s}(Y,s).$$

**Acknowledgments** This research was supported in part by the Humboldt Foundation’s Sofja Kovalevskaya Award.
2 Review of equivariant Seiberg-Witten Floer homology

In this section, we recall some of basic ingredients in the definition of the equivariant Seiberg-Witten Floer homology from [5] (See [5] for all the details).

Let \((Y, s)\) be a rational homology 3-sphere \(Y\) with a \(\text{Spin}^c\) structure \(s \in \text{Spin}^c(Y)\). For a good pair of metric and perturbation (a co-closed imaginary-valued 1-form \(\nu\)) on \(Y\), the 3-dimensional Seiberg-Witten equations on \((Y, s)\) (Cf. [2] [3] [4] [5]):

\[
\begin{align*}
* F_A &= \sigma(\psi, \psi) + \nu \\
\vartheta_A \psi &= 0,
\end{align*}
\]

(6)

for a pair of \(\text{Spin}^c\) connection \(A\) and a spinor \(\psi\), have only finitely many irreducible solutions (modulo the gauge transformations), denoted by \(\mathcal{M}_Y^+(s)\) the set of equivalence classes of irreducible solutions to (6), and \(\theta\) is the unique reducible solution (modulo the gauge transformations). Write \(\mathcal{M}_Y(s) = \mathcal{M}_Y^+(s) \cup \{\theta\}\).

Gauge classes of finite energy solutions to the 4-dimensional Seiberg-Witten equations, perturbed as in [2] [3] [5], can be regarded as moduli spaces of flowlines of the Chern-Simons-Dirac functional on the gauge equivalence classes of \(\text{Spin}^c\) connections and spinors for \((Y, s)\). These can be partitioned into moduli spaces of flowlines between pairs of critical points from \(\mathcal{M}_Y(s)\). Each is a smooth oriented manifold which can be compactified to a smooth manifold with corners by adding broken flowlines that split through intermediate critical points.

The spectral flow of the Hessian operator of the Chern-Simons-Dirac functional defines a relative grading on \(\mathcal{M}_Y(s)\):

\[
gr(\cdot, \cdot) : \mathcal{M}_Y(s) \times \mathcal{M}_Y(s) \to \mathbb{Z}.
\]

In particular, using the unique reducible point \(\theta\) in \(\mathcal{M}_Y(s)\), there is a \(\mathbb{Z}\)-lifting of the relative grading given by \(\text{gr}(a) = \text{gr}(a, \theta)\).

Let \(a\) be an irreducible monopole in \(\mathcal{M}_Y(s)\), then for any \(b \neq a\) in \(\mathcal{M}_Y(s)\), the moduli space of flowlines from \(a\) to \(b\), denoted by \(\mathcal{M}(a, b)\) has dimension \(\text{gr}(a) - \text{gr}(b) > 0\) (if non-empty). The moduli space of flowlines from \(\theta\) to \(d \in \mathcal{M}_Y^+(s)\), denoted by \(\mathcal{M}(\theta, d)\) has dimension \(-\text{gr}(d) - 1 > 0\) (if non-empty). Note that all these moduli spaces of flowlines admit an \(\mathbb{R}\)-action given by the \(\mathbb{R}\)-translation of flowlines: the corresponding quotient spaces are denoted by \(\widehat{\mathcal{M}}(a, b)\) and \(\widehat{\mathcal{M}}(\theta, d)\), respectively.
For any irreducible critical points $a$ and $c$ in $\mathcal{M}_Y(s)$ with $gr(a) - gr(c) = 2$, we can construct a canonical complex line bundle over $\mathcal{M}(a,c)$ and a canonical section as follows (see section 5.3 in [5]). Choose a base point $(y_0, t_0)$ on $Y \times \mathbb{R}$, and a complex line $\ell_{y_0}$ in the fiber $W_{y_0}$ of the spinor bundle $W$ over $y_0 \in Y$. We choose $\ell_{y_0}$ so that it does not contain the spinor part $\psi$ of any irreducible critical point. Since there are only finitely many critical points we can guarantee such choice exists. Denote the based moduli space of $\mathcal{M}(a,c)$ by $\mathcal{M}(O_a, O_c)$ as in [5], where $O_a$ and $O_c$ are the $U(1)$-orbits of monopoles on the based configuration space. We consider the line bundle

$$L_{ac} = \mathcal{M}(O_a, O_c) \times_{U(1)} (W_{y_0}/\ell_{y_0}) \to \mathcal{M}(a,c)$$  \hspace{1cm} (7)

with a section given by

$$s([A, \Psi]) = ([A, \Psi], \Psi(y_0, t_0)).$$  \hspace{1cm} (8)

For a generic choice of $(y_0, t_0)$ and $\ell_{y_0}$, the section $s$ of (8) has no zeroes on the boundary strata of the compactification of $\mathcal{M}(a,c)$. This determines a trivialization of $L_{ac}$ away from a compact set in $\mathcal{M}(a,c)$. The line bundle $L_{ac}$ over $\mathcal{M}(a,c)$, with the trivialization $\varphi$ specified above, has a well-defined relative Euler class (Cf. Lemma 5.7 in [5]).

**Definition 2.1.**

1. For any two irreducible critical points $a$ and $b$ in $\mathcal{M}_Y(s)$ with $gr(a) - gr(b) = 1$, we define $n_{ab} := \#(\mathcal{M}(a,b))$, the number of flowlines in $\mathcal{M}(a,b)$ counting with orientations. Similarly, for any $a \in \mathcal{M}_Y(s)$ with $gr(a) = 1$ and any $d \in \mathcal{M}_Y(s)$ with $gr(d) = -2$, we define $n_{a\theta} := \#(\mathcal{M}(a,\theta))$ and $n_{\theta d} := \#(\mathcal{M}(\theta,b))$, respectively.

2. For any two irreducible critical points $a$ and $c$ in $\mathcal{M}_Y(s)$ with $gr(a) - gr(c) = 1$, we define $n_{ac}$ to be the relative Euler number of the canonical line bundle $L_{ac}$ (7) with the canonical trivialization given by (8).

The following proposition states various relations satisfied by the integers defined in Definition 2.1, whose proof can be found in Remark 5.8 of [5].

**Proposition 2.2.**

1. For any irreducible critical point $a$ in $\mathcal{M}_Y^+(s)$ and any critical point $c$ in $\mathcal{M}_Y(s)$ with $gr(a) - gr(c) = 2$, we have the following identity:

$$\sum_{\substack{b \in \mathcal{M}_Y^+(s) \\text{gr}(a) - \text{gr}(b) = 1}} n_{ab} n_{bc} = 0.$$
2. Let $a$ and $d$ be two irreducible critical points with $\text{gr}(a) - \text{gr}(d) = 3$. Assume that all the critical points $c$ with $\text{gr}(a) > \text{gr}(c) > \text{gr}(d)$ are irreducible. Then we have the identity

$$
\sum_{c_1: \text{gr}(c_1) = 1} n_{a,c_1} m_{c_1,d} - \sum_{c_2: \text{gr}(c_2) = 1} m_{a,c_2} n_{c_2,d} = 0.
$$

When $\text{gr}(a) = 1$ and $\text{gr}(d) = -2$, we have the identity

$$
\sum_{c_1: \text{gr}(c_1) = 0} n_{a,c_1} m_{c_1,d} + n_{a,\theta} m_{\theta,d} - \sum_{c_2: \text{gr}(c_2) = -1} m_{a,c_2} n_{c_2,d} = 0.
$$

With the help of this Proposition, we can check that the equivariant Seiberg-Witten-Floer complex $CF_{s,U[1]}^{SW}(Y, s)$ as given in (1) with the grading and the differential operator given by (2) and (3) is well-defined, and we denote its homology by $HF_{s,U[1]}^{SW}(Y, s)$. The equivariant Seiberg-Witten-Floer cohomology, denoted by $HF_{s,U[1]}^{SW,*}(Y, s)$, is the homology of the dual complex $\text{Hom}(CF_{s,U[1]}^{SW}(Y, s), \mathbb{Z})$.

The main result in [5] shows that the equivariant Seiberg-Witten Floer homology $HF_{s,U[1]}^{SW}(Y, s)$ and cohomology $HF_{s,U[1]}^{SW,*}(Y, s)$ are topological invariants of $(Y, s)$.

3 Variants of equivariant Seiberg-Witten Floer homology

As mentioned in the introduction, we will generalize the construction of the equivariant Seiberg-Witten Floer homology in several ways.

First, we denote by $CF_{s,U[1]}^{SW,\infty}(Y, s)$ the graded complex generated by

$$
\{ \Omega^k \otimes \eta_a, \Omega^k \otimes 1_a, \Omega^k \otimes 1_\theta : a \in \mathcal{M}_y^*(s), k \in \mathbb{Z}. \}
$$

More precisely, for any irreducible critical orbits $O_a$, we set

$$
C_{s,U[1]}^{\infty}(O_a) = \mathbb{Z}[\Omega, \Omega^{-1}] \otimes \mathcal{O}_0(O_a)
:= \bigoplus_{k \in \mathbb{Z}} (\mathbb{Z} \Omega^k \otimes \eta_a + \mathbb{Z} \Omega^k \otimes 1_a)
$$

with the grading $\text{gr}(\Omega^k \otimes \eta_a) = 2k + \text{gr}(a)$ and $\text{gr}(\Omega^k \otimes 1_a) = 2k + \text{gr}(a) + 1$, and we set

$$
C_{s,U[1]}^{\infty}(\theta) = \bigoplus_{k \in \mathbb{Z}} \mathbb{Z} \Omega^k \otimes 1_\theta
$$

with the grading $\text{gr}(\Omega^k \otimes 1_\theta) = 2k$. 
We then consider
\[ \text{CF}^{SW,\infty}_{s,U[1]}(Y, s) = \bigoplus_{a \in M_Y(s)} \mathbb{Z}[\Omega, \Omega^{-1}] \otimes \Omega_0^s-\dim(O_a)(O_a), \]  
with the grading and the differential operator given by (2) and (3) respectively. That is, \( \text{CF}^{SW,\infty}_{s,U[1]}(Y, s) \)

is given by \[ \bigoplus_{a \in M_Y(s)} C^\infty_{s,U[1]}(O_a) \]
\[ = \bigoplus_{a \in M_Y(s)} C^\infty_{s,U[1]}(O_a) \oplus C^\infty_{s,U[1]}(\theta). \]

**Theorem 3.1.** Define \( H_{s,U[1]}^{SW,\infty}(Y, s) \) to be the homology of \( \text{CF}^{SW,\infty}_{s,U[1]}(Y, s), D \). Then we have
\[ H_{s,U[1]}^{SW,\infty}(Y, s) \cong \mathbb{Z}[\Omega, \Omega^{-1}], \]

**Proof.** Consider the filtration of \( \text{CF}^{SW,\infty}_{s,U[1]}(Y, s) \) according to the grading of the critical points
\[ \mathcal{F}_n \text{CF}^\infty_{s,U[1]} := \bigoplus_{\text{gr}(a) \leq n} C^\infty_{s,U[1]}(O_a) \]
the corresponding spectral sequence \( E^{\infty}_{kl} \). The filtration is exhaustive, that is,
\[ \text{CF}^{SW,\infty}_{s,U[1]}(Y, s) = \bigcup_n \mathcal{F}_n \text{CF}^\infty_{s,U[1]}, \]
and
\[ \cdots \subset \mathcal{F}_{n-1} \text{CF}^\infty_{s,U[1]} \subset \mathcal{F}_n \text{CF}^\infty_{s,U[1]} \subset \mathcal{F}_{n+1} \text{CF}^\infty_{s,U[1]} \subset \cdots \subset \text{CF}^{SW,\infty}_{s,U[1]}(Y, s). \]

Moreover, by the compactness of the moduli space of critical orbits, the set of indices \( \text{gr}(a) \) is bounded from above and below, hence the filtration is bounded. Thus, the spectral sequence converges to \( H_{s,U[1]}^{SW,\infty}(Y, s) \).

We compute the \( E^0 \)-term:
\[ E^0_{kl} = \mathcal{F}_k \text{CF}^\infty_{k+l,U[1]} \big/ \mathcal{F}_{k-1} \text{CF}^\infty_{k+l,U[1]} \]
\[ = \bigoplus_{a \in M_Y(s) : \text{gr}(a) = i \leq k} \mathcal{C}_k \text{CF}^\infty_{k+l-i,U[1]}(O_a) \big/ \bigoplus_{a \in M_Y(s) : \text{gr}(a) = i \leq k-1} \mathcal{C}_k \text{CF}^\infty_{k+l-i,U[1]}(O_a) \]
\[ = \bigoplus_{a \in M_Y(s) : \text{gr}(a) = k} \mathcal{C}_k \text{CF}^\infty_{l,U[1]}(O_a), \]
For $k \neq 0$ this complex is just the direct sum of the separate complexes $(C_{s,U(1)}^\infty(O_a), \partial_{U(1)})$ on each orbit $O_a$ with $\text{gr}(a) = k$:

$$\cdots \to \mathbb{Z} \cdot \Omega \otimes 1_a \xrightarrow{0} \mathbb{Z} \cdot \Omega \otimes \eta_a \xrightarrow{1} \mathbb{Z} \cdot 1 \otimes 1_a \xrightarrow{0} \mathbb{Z} \cdot 1 \otimes \eta_a \xrightarrow{1} \mathbb{Z} \cdot \Omega^{-1} \otimes 1_a \to \cdots$$  

(10)

In the case $k = 0$ we have

$$E_{0,t}^0 = C_{i,U(1)}^\infty(\emptyset) \oplus \bigoplus_{a \in \mathcal{M}_s^*(s) : \text{gr}(a) = 0} C_{i,U(1)}^\infty(O_a),$$

which again is a direct sum of the complexes $(C_{s,U(1)}^\infty(O_a), \partial_{U(1)})$, here $\partial_{U(1)}$ is the equivariant de Rham differential, and of the complex with generators $\Omega^r \otimes 1_\theta$ in degree $t = 2r$ and trivial differentials.

We then compute the $E_{pq}^1$ term directly: we have

$$E_{kl}^1 = H_{k+l}(E_{k,s}^0) = \begin{cases} 
\mathbb{Z} \cdot \Omega^r \otimes 1_\theta & k = 0, l = 2r \\
0 & k \neq 0,
\end{cases}$$

since each complex (10) is acyclic. Thus, the only non-trivial $E^1$-terms are of the form $E_{0,t}^1 = \mathbb{Z} \cdot \Omega^r \otimes 1_\theta$, $l = 2r$, with trivial differentials, so that the spectral sequence collapses and we obtain the result. \qed

### 3.1 Long exact sequence

**Definition 3.2.** Let $CF_{s,U(1)}^{SW,-}(Y,s)$ be the subcomplex of $CF_{s,U(1)}^{SW,\infty}(Y,s)$, generated by

$$\{\Omega^k \otimes \eta_a, \Omega^k \otimes 1_a, \Omega^k \otimes 1_\theta : a \in \mathcal{M}_s^*(s), k \in \mathbb{Z} \text{ and } k \neq 0\},$$

whose homology groups are denoted by $HF_{s,U(1)}^{SW,-}(Y,s)$. The quotient complex is denoted by $CF_{s,U(1)}^{SW,\infty}(Y,s)$, with the homology groups denoted by $HF_{s,U(1)}^{SW,\infty}(Y,s)$.

**Theorem 3.3.**

1. $HF_{s,U(1)}^{SW,\infty}(Y,s) \cong HF_{s,U(1)}^{SW}(Y,s)$, where $HF_{s,U(1)}^{SW}(Y,s)$ is the equivariant Seiberg-Witten-Floer homology defined in [5].

2. There is an exact sequence of $\mathbb{Z}$-modules which relates these variants of equivariant Seiberg-Witten-Floer homologies:

$$\cdots \to HF_{s,U(1)}^{SW,-}(Y,s) \xrightarrow{I_s} HF_{s,U(1)}^{SW,\infty}(Y,s) \xrightarrow{\pi_s} HF_{s,U(1)}^{SW,\infty}(Y,s) \xrightarrow{\delta_s} HF_{s-1,U(1)}^{SW,-}(Y,s) \to \cdots$$
Proof. It is easy to see that \( CF_{s,U(1)}^{SW,+}(Y, \theta) = CF_{s,U(1)}^{SW}(Y, \theta) \), with the same grading and differentials, hence \( HF_{s,U(1)}^{SW,+}(Y, \theta) \cong HF_{s,U(1)}^{SW}(Y, \theta) \). The long exact sequence in homology is induced by the short exact sequence of chain complexes:

\[
0 \to CF_{s,U(1)}^{SW,-}(Y, \theta) \to CF_{s,U(1)}^{SW,\infty}(Y, \theta) \to CF_{s,U(1)}^{SW,+}(Y, \theta) \to 0.
\]

\(\blacksquare\)

From the above long exact sequence, we can define

\[
HF_{red,s}^{SW}(Y, \theta) = \text{Coker}(\pi_s) \cong HF_{s,U(1)}^{SW,+}(Y, \theta)/\text{Ker}(\delta_s)
\]

\[
\cong \text{Im}(\delta_s) \cong \text{Ker}(I_{s-1}).
\]

(11)

3.2 The spectral sequence for \( HF_{s,U(1)}^{SW,+}(Y, \theta) \)

We consider again the filtration by index of critical orbits,

\[
\mathcal{F}_n C_{s,U(1)}^{+} := \bigoplus_{\text{gr}(a) \leq n} C_{s,U(1)}^{+}(O_a),
\]

for

\[
C_{s,U(1)}^{+}(O_a) = \mathbb{Z}[\Omega] \otimes \Omega_{0}^{s-\text{dim}(O_a)}(O_a).
\]

We have

\[
E^{0}_{kl} = \mathcal{F}_k C_{k+l,U(1)}^{+}/\mathcal{F}_{k-1} C_{k+l,U(1)}^{+}
\]

\[
= \bigoplus_{\text{gr}(a) = k} C_{l,U(1)}^{+}(O_a).
\]

This is a direct sum of the complexes

\[
\cdots \Rightarrow \mathbb{Z}, \Omega \otimes 1_a \Rightarrow \mathbb{Z}, \Omega \otimes \eta_a \Rightarrow \mathbb{Z}, 1 \otimes 1_a \Rightarrow \mathbb{Z}, 1 \otimes \eta_a \Rightarrow 0,
\]

(12)

over each orbit \( O_a \cong S^1 \) and, in the case \( k = 0 \), the complex with generators \( \Omega^r \otimes 1_\theta \) in degree \( l = 2r \), and trivial differentials.

Thus, we obtain that \( E^{1}_{pq} = H_{p+q}(E^{0}_{ps}) \) is of the form

\[
E^{1}_{pq} = \begin{cases} 
0 & q > 0 \\
\mathbb{Z}, 1 \otimes \eta_a & q = 0, \text{gr}(a) = k
\end{cases}
\]

10
for $k \neq 0$, and

$$E^1_{0q} = \begin{cases} 
\mathbb{Z} \Omega^r \otimes 1_{\theta} & q = 2r > 0 \\
\mathbb{Z} \cdot 1 \otimes \eta_a \oplus \mathbb{Z} \cdot 1_{\theta} & q = 0, \text{gr}(a) = 0.
\end{cases}$$

The differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is of the form

$$d^1(1 \otimes \eta_a) = n_{ab} 1 \otimes \eta_b$$

$$+ n_{a\theta} 1 \otimes 1_{\theta} \quad \text{(if \ gr}(a) = 1)$$

Thus, we obtain

$$E^2_{pq} = \begin{cases} 
HF^S_p(Y, s) & p \neq 0, q = 0 \\
\text{Ker}(\Delta_1) & p = 1, q = 0 \\
HF^S_0(Y, s) \oplus T_0 & p = 0, q = 0 \\
\mathbb{Z} \cdot \Omega^r \otimes 1_{\theta} & p = 0, q = 2r > 0.
\end{cases}$$

Here $HF^S_p(Y, s)$ denotes the non-equivariant (metric and perturbation dependent) Seiberg–Witten Floer homology. This is the homology of the complex with generators $1 \otimes \eta_a$ in degree $\text{gr}(a)$ and boundary coefficients $n_{ab}$ for $\text{gr}(a) - \text{gr}(b) = 1$. We also denoted by $\Delta_1$ the map

$$\Delta_1 : HF^S_1(Y, s) \to \mathbb{Z} \cdot 1 \otimes 1_{\theta},$$

$$\Delta_1(\sum x_a 1 \otimes \eta_a) = \sum x_a n_{a\theta} 1 \otimes 1_{\theta},$$

where the coefficients $x_a$ satisfy $\sum x_a n_{ab} = 0$. Finally, the term $T_0$ denotes the term

$$T_0 = \mathbb{Z} \cdot 1 \otimes 1_{\theta} / \text{Im}(\Delta_1).$$

Notice then that the boundary $d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$ is trivial, hence the $E^3$ terms are disposed as in the diagram:
The differential $d^3 : E^3_{p,q} \to E^3_{p-3,q+2}$ is given by the expression

$$d^3([\sum x_a 1 \otimes \eta_a]) = \sum x_a m_{a_0 \ldots a_{q+2}} \Omega \otimes 1_{\theta}, \quad (13)$$

for $\text{gr}(a) - \text{gr}(c) = 2$. The expression is obtained by considering the unique choice of a representative of the class $[\sum x_a 1 \otimes \eta_a]$ in $E^3_{p,q}$ whose boundary (3) defines a class in $E^3_{p-3,q+2}$.

The differential $d^1 : E^1_{p,q} \to E^1_{p-4,q+3}$ is again trivial, and we obtain the $E^5_{pq}$ of the form

\[
\begin{array}{cccccccccccc}
\cdots & 0 & 0 & 0 & 0 & \mathbb{Z} \cdot \Omega^2 \otimes 1_{\theta} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & T_1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\end{array}
\]

where again we denote by $T_1$ the term

$$T_1 = \mathbb{Z} \cdot \Omega \otimes 1_{\theta} / \text{Im}(\Delta_3).$$

Thus, by iterating the process, we observe that all the differentials $d^{2k} : E^{2k}_{p,q} \to E^{2k}_{p-2k,q+2k+1}$ are trivial and the differentials $d^{2k+1} : E^{2k+1}_{p,q} \to E^{2k+1}_{p-2k-1,q+2k}$ consists of one map for $p = 2k + 1$, $q = 0$:

$$\Delta_{2k+1} : HF^SW_{2k+1} \to \mathbb{Z} \cdot \Omega^k \otimes 1_{\theta},$$

induced by

$$\Delta_{2k+1}(\sum x_a 1 \otimes \eta_a) = \sum x_a m_{a_0 \ldots a_{2k-1} a_{2k-1} a_{2k-3} \ldots a_3 a_0} \Omega \otimes 1_{\theta}.$$ 

Here we have $\text{gr}(a) = 2k + 1$ and $\text{gr}(ar) = r$. Notice that these maps agree with the morphism $\Delta_s$, which is obtained in [5] as the connecting homomorphism in the long exact sequence relating equivariant and non-equivariant Seiberg–Witten Floer homologies.

We thus obtain the following structure theorem for equivariant Seiberg–Witten Floer homology.
**Theorem 3.4.** The equivariant Seiberg–Witten Floer homology \( HF_{s,U(1)}^{SW,\pm}(Y, s) \) has the form

\[
HF_{s,U(1)}^{SW,\pm}(Y, s) = \begin{cases} 
\text{Ker}(\Delta_{2k+1}) & * = 2k + 1 > 0 \\
HF_{2k}^{SW}(Y, s) \oplus T_k & * = 2k \geq 0 \\
HF_{s}^{SW}(Y, s) & * < 0
\end{cases}
\]

where \( T_k \) is the term

\[
T_k = \mathbb{Z}, \Omega^k \otimes 1_0/\text{Im}(\Delta_{2k+1}).
\]

This result refines the long exact sequence obtained in [5]:

\[
\xymatrix@=20pt{HF_{s,U(1)}^{SW}(Y, s) \ar[r]^-{i_*} \ar[d]_-{j_*} & HF_{s}^{SW}(Y, s, g, \nu) \ar[d]^-{\Delta_*} \\
\mathbb{Z}[\Omega] & \mathbb{Z}[\Omega].}
\]

Similar results can be obtained for \( HF_{s,U(1)}^{SW,-}(Y, s) \).

**3.3 Topological invariance**

Note that the definitions of these homologies depend on the Seiberg-Witten equations, which use the metric and perturbation on \((Y, s)\). By the result of [5], we know that \( HF_{s,U(1)}^{SW,\pm}(Y, s) \cong HF_{s,U(1)}^{SW}(Y, s) \) is a topological invariant of \((Y, s)\), we first recall this topological invariance as stated in Theorem 6.1 [5].

**Theorem 3.5.** (Theorem 6.1 [5]) Let \((Y, s)\) be a rational homology sphere with a Spin \( c \) structure. Suppose given two metrics \( g_0 \) and \( g_1 \) on \( Y \) and perturbations \( \nu_0 \) and \( \nu_1 \) such that \( \text{Ker}(\phi^0_{U(1)}) = \text{Ker}(\phi^1_{U(1)}) = 0 \), so that the corresponding monopole moduli spaces \( M_Y(s, g_0, \nu_0) \) and \( M_Y(s, g_1, \nu_1) \) consist of finitely many isolated points. Then there exists an isomorphism between the equivariant Seiberg-Witten Floer homologies \( HF_{s,U(1)}^{SW}(Y, s, g_0, \nu_0) \) and \( HF_{s,U(1)}^{SW}(Y, s, g_1, \nu_1) \), with a degree shift given by the spectral flow of the Dirac operator \( \phi^0_{U(1)} \) along a path of metrics and perturbations connecting \((g_0, \nu_0)\) and \((g_1, \nu_1)\).

That is, if the complex spectral flow along the path \((g_t, \nu_t)\) is denoted by \( SF_{\mathbb{C}}(\phi^0_{U(1)}) \), then for any \( k \in \mathbb{Z} \),

\[
HF_{k,U(1)}^{SW}(Y, s, g_0, \nu_0) \cong HF_{k+2SF_{\mathbb{C}}(\phi^0_{U(1)}),U(1)}^{SW}(Y, s, g_1, \nu_1).
\]

From Theorem 3.1, we know that

\[
HF_{s,U(1)}^{SW,\infty}(Y, s) \cong \mathbb{Z}[\Omega, \Omega^{-1}]
\]
is independent of \((Y, s)\), up to a degree shift as given in Theorem 3.5. Thus, applying the five lemma to the long exact sequence in Theorem 3.3, we obtain that \(HF_{s, U(1)}^{SW, -}(Y, s)\) and \(HF_{red, s}^{SW}(Y, s)\) are also topological invariants of \((Y, s)\).

**Theorem 3.6.** \(HF_{s, U(1)}^{SW, -}(Y, s)\) and \(HF_{red, s}^{SW}(Y, s)\) are topological invariants of \((Y, s)\), in the sense that, given any two metrics \(g_0\) and \(g_1\) on \(Y\) and perturbations \(\nu_0\) and \(\nu_1\), with \(\text{Ker}(\varphi_{g_0}^{\theta}) = \text{Ker}(\varphi_{g_1}^{\theta}) = 0\), there exist isomorphisms

\[
HF_{k, U(1)}^{SW, -}(Y, s, g_0, \nu_0) \cong HF_{k+2S_{C}(\varphi_{g_0}^{\theta}), U(1)}^{SW, -}(Y, s, g_1, \nu_1)
\]

\[
HF_{red, k}^{SW}(Y, s, g_0, \nu_0) \cong HF_{red, k+2S_{C}(\varphi_{g_0}^{\theta})}^{SW}(Y, s, g_1, \nu_1).
\]

Here \(S_{C}(\varphi_{g}^{\theta})\) denotes the complex spectral flow of the Dirac operator \(\varphi_{g}^{\theta}\) along the path \((g, \nu_t)\).

4 Properties of equivariant Seiberg-Witten Floer homologies

In this section, we briefly discuss some of the algebraic structures and properties of the equivariant Seiberg-Witten Floer homologies defined in the previous section.

Note that for any irreducible critical points \(a\) and \(b\) in \(M_g^*(s)\), the associated integer \(m_{ac}\) is the counting of points in the geometric representative of the relative first Chern class of the canonical line bundle (7) over \(M(a, c)\), we can apply this fact to define a \(u\)-action on the chain complex \(CF_{s, U(1)}^{SW, \infty}(Y, s)\)

\[
u : CF_{s, U(1)}^{SW, \infty}(Y, s) \longrightarrow CF_{s, U(1)}^{SW, \infty}(Y, s)
\]

which decreases the grading by two. The action is given in terms of its actions on generators as follows:

\[
u(\Omega^n \otimes \eta_a) = \sum_{c \in \mathcal{M}_g^*(Y, s)} m_{ac} \Omega^n \otimes \eta_c.
\]

\[
u(\Omega^n \otimes 1_a) = \begin{cases} 
\sum_{c \in \mathcal{M}_g^*(Y, s)} m_{ac} \Omega^n \otimes 1_c & \text{if } gr(a) \neq 1 \\
\sum_{c \in \mathcal{M}_g^*(Y, s)} m_{ac} \Omega^n \otimes 1_c + n_{a\theta} \Omega^n \otimes 1_\theta & \text{if } gr(a) = 1
\end{cases}
\]

(14)

\[
u(\Omega^n \otimes 1_\theta) = \sum_{d \in \mathcal{M}_g^*(s)} n_{a\theta d} \Omega^n \times \eta_d + \Omega^{n-1} \otimes 1_\theta.
\]
Proposition 4.1. The $u$-action defined (14) on the chain complex $CF_{*,U[1]}^{SW,\infty}(Y,s)$ is homotopic to the $\Omega^{-1}$-action acting on $CF_{*,U[1]}^{SW,\infty}(Y,s)$. The induced actions on $CF_{*,U[1]}^{SW,\pm}(Y,s)$ define $\mathbb{Z}[u]$-module structures on $HF_{*,U[1]}^{SW,\pm}(Y,s)$.

Proof. Define $H : CF_{*,U[1]}^{SW,\infty}(Y,s) \to CF_{*,U[1]}^{SW,\infty}(Y,s)$ by its actions on the generators as follows:

$$H(\Omega^n \otimes \eta_a) = 0,$$
$$H(\Omega^n \otimes 1_a) = \Omega^n \otimes \eta_a,$$
$$H(\Omega^n \otimes 1_\theta) = 0.$$

Then it is a direct calculation to show that we have:

$$(u - \Omega^{-1})(\Omega^k \otimes \eta_a) = m_{ac}\Omega^k \otimes \eta_c - \Omega^{k-1} \otimes \eta_a = (DH + HD)(\Omega^k \otimes \eta_a)$$

$$(u - \Omega^{-1})(\Omega^k \otimes 1_a) = m_{ac}\Omega^k \otimes 1_c - \Omega^{k-1} \otimes 1_a + n_{a\theta}\Omega^n \otimes 1_\theta = (DH + HD)(\Omega^k \otimes 1_a),$$

$$(u - \Omega^{-1})(\Omega^k \otimes 1_\theta) = n_{\theta d}\Omega^n \otimes \eta_d = (DH + HD)(\Omega^k \otimes 1_\theta).$$

Thus the claim follows using the chain homotopy $u - \Omega^{-1} = D \circ H + H \circ D$.

Thus, on the homological level, we can identify the $u$-action with the induced $\Omega^{-1}$ action on various homologies. In particular, we see that there is a subcomplex $\overline{CF}^{SW}_{*}(Y,s)$ of $CF_{*,U[1]}^{SW}(Y,s)$ such that the following short exact sequence of chain complexes holds:

$$0 \to \overline{CF}^{SW}_{*}(Y,s) \to CF_{*,U[1]}^{SW}(Y,s) \overset{\Omega^{-1}}{\to} CF_{*,U[1]}^{SW,\pm}(Y,s) \to 0. \quad (15)$$

Proposition 4.2. Let $HF_{*,U[1]}^{SW}(Y,s)$ be the homology of $\overline{CF}^{SW}_{*}(Y,s)$, then $HF_{*,U[1]}^{SW}(Y,s)$ is also a topological invariant of $(Y,s)$, and it is determined by the following long exact sequence

$$\cdots \to HF_{*,U[1]}^{SW}(Y,s) \xrightarrow{u} HF_{*,U[1]}^{SW,\pm}(Y,s) \to HF_{*,U[1]}^{SW,\pm}(Y,s) \to \cdots.$$

Moreover, $HF_{*,U[1]}^{SW}(Y,s)$ is non-trivial if and only if $HF_{*,U[1]}^{SW,\pm}(Y,s)$ is non-trivial.

Proof. The long exact sequence follows from the short exact sequence of chain complexes (15) and Proposition 4.1. This long exact sequence implies that $HF_{*,U[1]}^{SW}(Y,s)$ is also a topological invariant of $(Y,s)$.
Note that, from the compactness of $\mathcal{M}_Y(s)$, we see that each element in $HF_{s, U[1]}^{SW, +}(Y, s)$ can be annihilated by a sufficiently large power of $\Omega^{-1}$. Hence, $u$ is an isomorphism on $HF_{s, U[1]}^{SW, +}(Y, s)$ if and only if $HF_{s, U[1]}^{SW, +}(Y, s)$ is trivial. Then the last claim follows from this observation and the long exact sequence. □

If we think of the set of Spin$^c$ structures on $Y$ as the set of equivalence classes of nowhere vanishing vector fields on $Y$ (Cf.[9]), then there is a natural bijection between Spin$^c(Y)$ and Spin$^c(\neg Y)$ where $\neg Y$ is the same $Y$ with the opposite orientation.

**Theorem 4.3.** Let $(Y, s)$ be a rational homology 3-sphere with a Spin$^c$ structure $s$, and $(\neg Y, s)$ denote $Y$ with the opposite orientation and the corresponding Spin$^c$ structure. Then there is a natural isomorphism

$$HF_{U[1]}^{SW, s}(Y, s) \cong HF_{U[1]}^{SW, -}(\neg Y, s)$$

where $HF_{U[1]}^{SW, s}(Y, s)$ is the equivariant Seiberg-Witten-Floer cohomology defined in [5].

**Proof.** Note that $HF_{U[1]}^{SW, s}(Y, s)$ is the homology of the dual complex $\text{Hom}(CF_{s, U[1]}^{SW, +}(Y, s), \mathbb{Z})$. We start to construct a natural pairing

$$\langle \cdot, \cdot \rangle : \quad CF_{s, U[1]}^{SW, \infty}(Y, s) \times CF_{s, U[1]}^{SW, \infty}(\neg Y, s) \rightarrow \mathbb{Z} \quad (16)$$

which satisfies

$$\langle D_Y(\xi_1), \xi_2 \rangle = \langle \xi_1, D_{\neg Y}(\xi_2) \rangle, \quad \langle \Omega^{-1}(\xi_1), \xi_2 \rangle = \langle \xi_1, \Omega^{-1}(\xi_2) \rangle. \quad (17)$$

for any element $\xi_1 \in CF_{s, U[1]}^{SW, \infty}(Y, s)$ and any element $\xi_2 \in CF_{s, U[1]}^{SW, \infty}(\neg Y, s)$.

Then we will show that the above pairing is non-degenerate when restricted to $CF_{s, U[1]}^{SW, +}(Y, s) \times CF_{s, U[1]}^{SW, -}(\neg Y, s)$.

From the nature of the Seiberg-Witten equations, we see that there is an identification

$$\mathcal{M}_Y(s) \rightarrow \mathcal{M}_Y(\neg s)$$

for a good pair of metric and perturbation on $(Y, s)$ and the corresponding metric and perturbation on $(\neg Y, s)$. Then the relative gradings with respect to the unique reducible monopole in $\mathcal{M}_Y(s)$ and
\[ M_{-Y}(s) \text{ respectively, satisfies} \]

\[ gr_{-Y}(a^{-}) = -gr_Y(a) - 1, \]

where \( a^{-} \) is the element in \( M_{-Y}(s) \) corresponding to \( a \in M_{+Y}(s) \), we assume that \( gr_{Y}(\theta) = gr_{-Y}(\theta^{-}) \).

Moreover, there is a natural identification between the moduli spaces of flowlines for \((Y, s)\) and \((-Y, s)\), that is,

\[ M_{Y \times \mathbb{R}}(a, b) \cong M_{-Y \times \mathbb{R}}(b^{-}, a^{-}). \]

Now we define the pairing on \( CF_{*U(1)}^{SW, \infty}(Y, s) \times CF_{*U(1)}^{SW, \infty}(-Y, s) \) such that the following pairings are the only non-trivial pairings:

\[ \langle \Omega^n \otimes \eta_a, \Omega^{-n-1} \otimes 1_{a^{-}} \rangle = 1 \]

\[ \langle \Omega^n \otimes 1_a, \Omega^{-n-1} \otimes \eta_{a^{-}} \rangle = 1 \]

\[ \langle \Omega^n \otimes 1_{\theta}, \Omega^{-n-1} \otimes 1_{\theta^{-}} \rangle = 1. \]

It is a direct calculation to show that this pairing satisfies the relation (17) and the restriction of this pairing to \( CF_{*U(1)}^{SW, +}(Y, s) \times CF_{*U(1)}^{SW, -}(-Y, s) \) is non-degenerate. Then the claim follows from the definition.

Let \( \overline{HF}_{*U(1)}^{SW}(Y, s) \) and \( \overline{HF}_{*U(1)}^{SW, \pm}(Y, s) \) denote the homology groups of the dual complexes \( \text{Hom}(\overline{CF}_{*U(1)}^{SW}(Y, s), \mathbb{Z}) \) and \( \text{Hom}(CF_{*U(1)}^{SW, \pm}(Y, s), \mathbb{Z}) \) of \( \overline{CF}_{*U(1)}^{SW}(Y, s) \) and \( CF_{*U(1)}^{SW, \pm}(Y, s) \) respectively. From the proof of the above Theorem 4.3, we actually establish the following duality between these homologies.

**Theorem 4.4.** For any rational homology 3-sphere \( Y \) with a spinc structure \( s \), there exist natural isomorphisms

\[ \overline{HF}_{*U(1)}^{SW}(Y, s) \cong \overline{HF}_{*U(1)}^{SW}(-Y, s), \quad \overline{HF}_{*U(1)}^{SW, \pm}(Y, s) \cong \overline{HF}_{*U(1)}^{SW, \pm}(-Y, s). \]
References


Matilde Marcolli and Bai-Ling Wang,
Max–Planck–Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany.
marcolli@mpim-bonn.mpg.de
bwang@mpim-bonn.mpg.de