

## NONCOMMUTATIVE GEOMETRY AND NUMBER THEORY

Noncommutative geometry is a modern field of mathematics created by Alain Connes at the beginning of the eighties. It provides powerful tools to treat spaces that are essentially of a quantum nature. Unlike the case of ordinary spaces, their algebra of coordinates is noncommutative, reflecting phenomena like the Heisenberg uncertainty principle in quantum mechanics.

What is especially interesting is the fact that such quantum spaces are abundant in mathematics. One obtains them easily when one considers equivalence relations which are so drastic that they tend to collapse most points together, yet one wishes to retain enough information in the process to be able to do interesting geometry on the resulting space.

In such cases, noncommutative geometry shows that there is a quantum cloud surrounding the classical space, which retains all the essential geometric information, even when the underlying classical space becomes extremely degenerate. It is to this quantum aura that all sophisticated tools of geometry and mathematical analysis, properly reinterpreted, can still be applied.

It became increasingly evident in recent years that the tools of noncommutative geometry may find new and important applications to Number Theory, a very different branch of pure mathematics with an ancient and illustrious history. This happened mostly through a new approach of Connes to the Riemann hypothesis (at present the most famous unsolved problem in mathematics).

The first instance of such connections between noncommutative geometry and number theory emerged earlier, when Bost and Connes discovered a very interesting noncommutative space with remarkable arithmetic properties. The system it describes consists of quantized optical phases, discretized at different scales. These are essentially the phasors used in modeling quantum computers (*cf.* Figure 1). A mechanism that accounts for consistency over scale changes organizes the phasors via a kind of renormalization procedure. This consistency condition imposes the equivalence relation that makes the resulting space noncommutative.

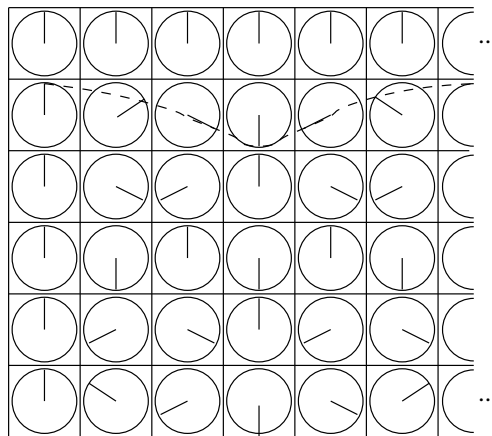


FIGURE 1. Phase operators:  $\mathbb{Z}/6\mathbb{Z}$  discretization.

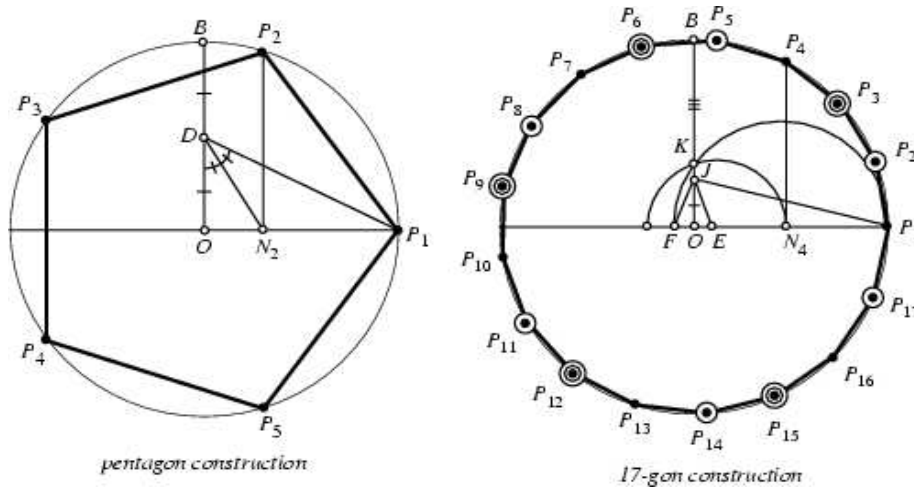


FIGURE 2. Construction of polygons by ruler and compass

The system obtained this way has an intrinsic dynamics which makes it evolve in time and one can consider corresponding thermodynamical equilibrium states at various temperatures. Above a certain critical temperature the distribution of phases is essentially chaotic and there is a unique equilibrium state. At the critical temperature the system undergoes a phase transition with spontaneous symmetry breaking and below critical temperature the system exhibits many different equilibrium states parameterized by arithmetic data.

Especially interesting is what happens at zero temperature. There the arithmetic structure that governs the action of the symmetry group of the system on the extremal ground states is the same one that answers the famous mathematical problem (solved by Gauss) of which regular polygons can be constructed using only ruler and compass (*cf.* Figure 2).

The crucial feature that allows for a solution of this geometric problem is the fact that in addition to the obvious rotational symmetries of regular polygons there exists another hidden and much more subtle symmetry coming from the Galois group  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , a very beautiful and still mysterious object, which in this case manifests itself not through the multiplicative action of roots of unity (rotations of the vertices of the polygons) but through the operation of raising them to powers.

Thus, from the example of the Bost Connes noncommutative space, a dictionary emerges that relates the phenomena of spontaneous symmetry breaking in quantum statistical mechanics to the mathematics of Galois theory. Moreover, the partition function of this quantum statistical mechanical system is an object of central interest in Number Theory, namely the Riemann zeta function (*cf.* Figure 3).

More recently, other results that point to deep connections between noncommutative geometry and number theory appeared in the work of Connes and Moscovici on the modular Hecke algebras which shows that the Rankin–Cohen brackets, an important algebraic structure on modular forms extensively studied years ago by Zagier (at MPIM), have a natural interpretation in the language of noncommutative geometry. Modular forms are a very important class of functions that plays a fundamental role in many fields of mathematics, especially in Number Theory and Arithmetic Geometry. They exhibit very elaborate symmetry patterns associated to certain tessellations of the hyperbolic plane (*cf.* Figure 4).

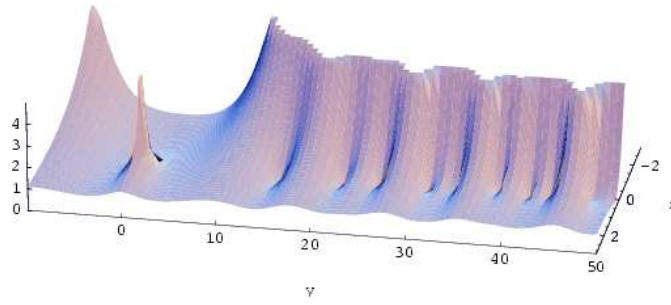


FIGURE 3. The Riemann zeta function

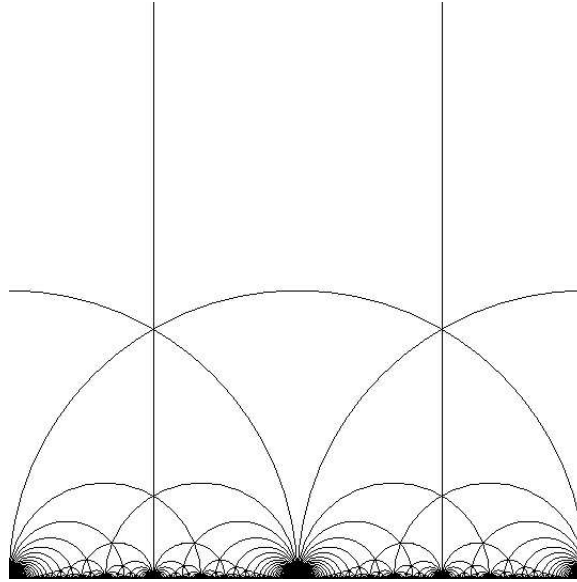


FIGURE 4. The modular curve

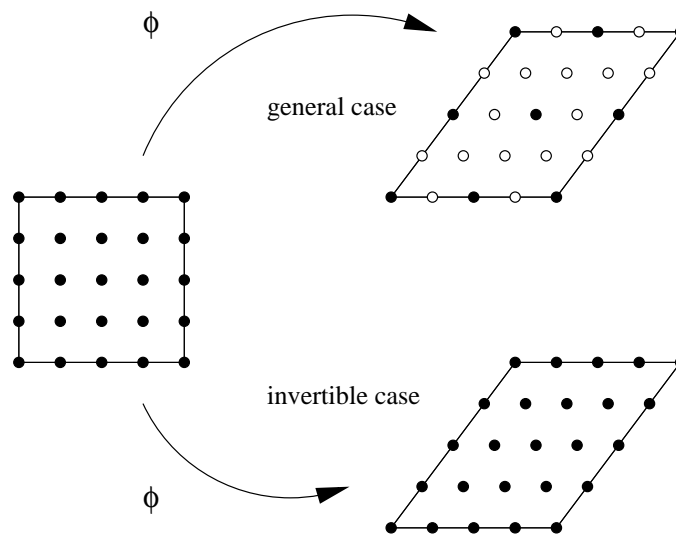
When viewed with the eyes of noncommutative geometry the algebraic structures studied by Zagier appear as a manifestation of a type of symmetry of noncommutative spaces, related to the transverse geometry of codimension one foliations (Figure 5), which was investigated extensively in work of Connes and Moscovici.

The special tessellations of the hyperbolic plane mentioned in relation to modular forms give rise to a family of 2-dimensional surfaces known as the modular curves. Recent work of Manin and Marcolli (both at MPIM) showed that much of the rich arithmetic structure of the modular curves is captured by a noncommutative space, which arises from the tessellation restricted to the infinitely distant horizon of the hyperbolic plane (the bottom horizontal line in Figure 4). The fact that the infinite horizon of modular curves hides a noncommutative space was also observed in work of Connes, Douglas and Schwarz in the context of String Theory.

Ongoing work of Connes and Marcolli (at MPIM) uncovered the remarkable fact that all the instances listed above of interactions between number theory and noncommutative geometry (Connes' work on the Riemann zeta function, the Bost–Connes system, the modular Hecke algebra, the noncommutative boundary of modular curves) are in fact manifestations of



FIGURE 5. An example of codimension one foliations

FIGURE 6.  $\mathbb{Q}$ -lattices

one and the same underlying noncommutative space, namely the space of commensurability classes of  $\mathbb{Q}$ -lattices.

A lattice consists of arrays of points in a vector space, disposed like atoms in a crystal. For example, the set of points with integer coordinates in the plane is a 2-dimensional lattice. A  $\mathbb{Q}$ -lattice is one such object where one has a way of labeling the points of rational coordinates inside the fundamental cell of the lattice. If each rational point is labeled in a unique way the  $\mathbb{Q}$ -lattice is said to be invertible, while in general one also allows for labelings that miss certain arrays of points while assigning multiple labels to others (*cf.* Figure 6).

When studying the geometric properties of  $\mathbb{Q}$ -lattices, it is natural to treat as the same object all  $\mathbb{Q}$ -lattices that have the same rational points and where the respective labelings agree whenever both are defined. This determines an equivalence relation on the set of  $\mathbb{Q}$ -lattices.

One observes that the identifications produced by this seemingly harmless equivalence relation are in fact drastic enough to give rise to a noncommutative space. On the other hand, if one restricts the attention to just invertible  $\mathbb{Q}$ -lattices, these are organized in a classical space. In the case of 2-dimensional lattices, the parameterizing space is the family of all modular curves.

Since  $\mathbb{Q}$ -lattices exist in any dimension, there is in any dimension a corresponding noncommutative space. The Bost–Connes space is just the space of commensurability classes of 1-dimensional  $\mathbb{Q}$ -lattices considered up to a scaling factor. The noncommutative space introduced by Connes in the spectral realization of the zeros of the Riemann zeta function (whose position in the plane is the content of the Riemann hypothesis) is the space of commensurability classes of 1-dimensional  $\mathbb{Q}$ -lattices with the scale factor taken into account. The modular Hecke algebra of Connes and Moscovici is a piece of the algebra of coordinates on the space of commensurability classes of 2-dimensional  $\mathbb{Q}$ -lattices and the noncommutative boundary of modular curves is a stratum in this space that accounts for possible degenerations of the 2-dimensional lattice.

The noncommutative space of commensurability classes of 2-dimensional  $\mathbb{Q}$ -lattices up to scale also has a natural time evolution and one can investigate the structure of the corresponding thermodynamical equilibria. At zero temperature this quantum space freezes on the underlying classical space (the family of modular curves) and all quantum fluctuations cease. The extremal states at zero temperature correspond to points on a modular curve. When the temperature rises quantum effects become predominant and the system undergoes a first phase transition where all the different equilibrium states merge, leaving a unique chaotic state. There is then a second critical temperature where the system experiences another phase transition after which no equilibrium state survives.

What acts as group of symmetries of this quantum mechanical system is the group of all arithmetic symmetries of the modular functions. As in the 1-dimensional case, the induced action on the extremal states at zero temperature is via Galois theory. In this 2-dimensional system, however, not all symmetries act directly on the classical space at zero temperature as they need the more refined structure of the quantum system, hence one obtains the Galois action at zero temperature by warming up below critical temperature, looking at the full symmetries of the quantum system, and then cooling down again to zero temperature where arithmeticity becomes apparent.

The noncommutative space of commensurability classes of  $\mathbb{Q}$ -lattices with its rich arithmetic structure provides a valuable tool for investigating many related number theoretic questions. For instance, in the spectral realization of zeros of the Riemann zeta function an important question is how to pass consistently to extensions of the field of rational numbers. Dealing with such questions will involve the noncommutative spaces of higher dimensional  $\mathbb{Q}$ -lattices and their interrelations.