Holography principle and arithmetic of algebraic curves

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<u>Arithmetic surfaces</u>

Projective algebraic curve X defined over $\mathbb{Q}$; equations w/ $\mathbb{Z}$-coefficients $\Rightarrow$ scheme $X_{\mathbb{Z}}$; closed fiber of $X_{\mathbb{Z}}$ at $p \in \mathrm{Spec}(\mathbb{Z})$ reduction $X_{\mathbb{Z}} \mod p$

infinitesimal neighborhoods: reductions of $X_{\mathbb{Z}}$ mod p^n. Limit $n \to \infty$: p–adic completion of $X_{\mathbb{Z}}$

arithmetic infinity: embedding $\mathbb{Q} \to \mathbb{C}$ (absolute value vs. p-adic valuations)

Arakelov: Hermitian geometry of $X_{\mathbb{C}}$ analog of p–adic completions of $X_{\mathbb{Z}}$: **Green functions** provide intersection indices of arithmetic curves over the fiber at infinity.

Green function

compact Riemann surface $X_{\mathbb{C}}$, Green function $g_{\mu,A}$: divisor $A = \sum_x m_x(x)$, positive real–analytic 2-form $d\mu$

- *Laplace equation*: g_A satisfies $\partial\bar\partial\, g_A = \pi i\,(\deg(A)\, d\mu - \delta_A)$, with δ_A the δ–current $\varphi \mapsto \sum_x m_x \varphi(x)$.

- *Singularities*: $z =$ loc coord in neighb of $x \Rightarrow g_A - m_x \log|z|$ loc real analytic.

- *Normalization*: g_A satisfies $\int_X g_A d\mu = 0$.

$B = \sum_y n_y(y)$ divisor, $|A| \cap |B| = \emptyset$,

$g_\mu(A, B) := \sum_y n_y g_{\mu,A}(y)$ symmetric biadditive

g_μ depends on μ. In case of degree zero divisors, $\deg A = \deg B = 0$, $g_\mu(A, B) = g(A, B)$ conformal invariant.

If $A = Div(w_A)$, w_A meromorphic function

$$g(A,B) = \log \prod_{y \in |B|} |w_A(y)|^{n_y}$$

$\mathbb{P}^1(\mathbb{C})$: in terms of cross-ratio $a,b,c,d \in \mathbb{P}^1(\mathbb{C})$:

$$\log |\langle a,b,c,d \rangle|$$

$\langle a,b,c,d \rangle = \frac{(a-b)(c-d)}{(a-d)(c-b)}$

General case: A, B degree zero div on $X_{\mathbb{C}}$, ω_A differential of third kind w/ purely imaginary periods and residues m_x at x: Green function

$$g(A,B) = Re \int_{\gamma_B} \omega_A,$$

γ_B = 1–chain with boundary B

$\Rightarrow$ basis of differentials of third kind on $X_{\mathbb{C}}$

Space–time

Anti de Sitter: AdS_{d+1} space–time satisfying Einstein's eq w/constant curvature $R < 0$ (empty space with negative cosmological constant)

In general relativity: $\mathrm{AdS}_{3+1} = S^1 \times \mathbb{R}^3$, metrically hyperboloid $-u^2 - v^2 + x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^5$, $ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$.

To avoid time–like closed geodesics $\Rightarrow$ univ cover $\widetilde{\mathrm{AdS}_{3+1}}$ topologically $\mathbb{R}^4$. Boundary at infinity of $\widetilde{\mathrm{AdS}_{d+1}}$ is a compactification of d–dimensional Minkowsky space–time

Euclidean signature $\mathrm{AdS}_{d+1} \Rightarrow \mathbb{H}^{d+1}$ real hyperbolic space

In **quantum gravity** AdS_{2+1} and $\mathbb{H}^3$

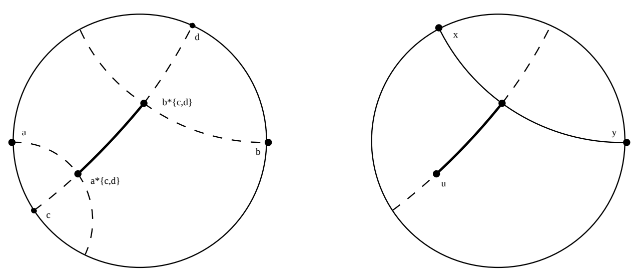
The Holography principle

Bulk space (asymptotically AdS) and Boundary (conformal boundary at infinity): Gravity on bulk space $\leftrightsquigarrow$ field theory on the boundary
$2+1$: Geodesic propagator on bulk $\leftrightsquigarrow$ bosonic propagator on the boundary (for Riemann surfaces: Boson/Fermion equivalence)

Genus zero case Euclidean signature: $\mathbb{H}^3 \leftrightsquigarrow \mathbb{P}^1(\mathbb{C})$:

Boundary propagator (i.e. Green function) **= geodesic propagator on the bulk space**

$$g((a)-(b),(c)-(d)) = -\text{ordist}\ (a*\{c,d\}, b*\{c,d\})$$



The propagators: $a \to c$, $b \to d$, logarithmic divergence: intrinsic, no choice of cut-off functions (cf.Balasubramanian-Ross Phys.Rev.D 3 61 (2000) 4)

Bañados–Teitelboim–Zanelli black hole

Genus one case: $\mathbb{H}^3/(q^{\mathbb{Z}}) \leftrightsquigarrow X_q(\mathbb{C}) = \mathbb{C}^*/(q^{\mathbb{Z}})$

(Jacobi uniformization) $q : (z, y) \mapsto (qz, |q|y)$

$$q = \exp\left(\frac{2\pi(i|r_-| - r_+)}{\ell}\right)$$

$$r_\pm^2 = \frac{1}{2}\left(M\ell \pm \sqrt{M^2\ell^2 + J}\right)$$

mass and angular momentum of black hole, $-1/\ell^2$ = cosmological constant.

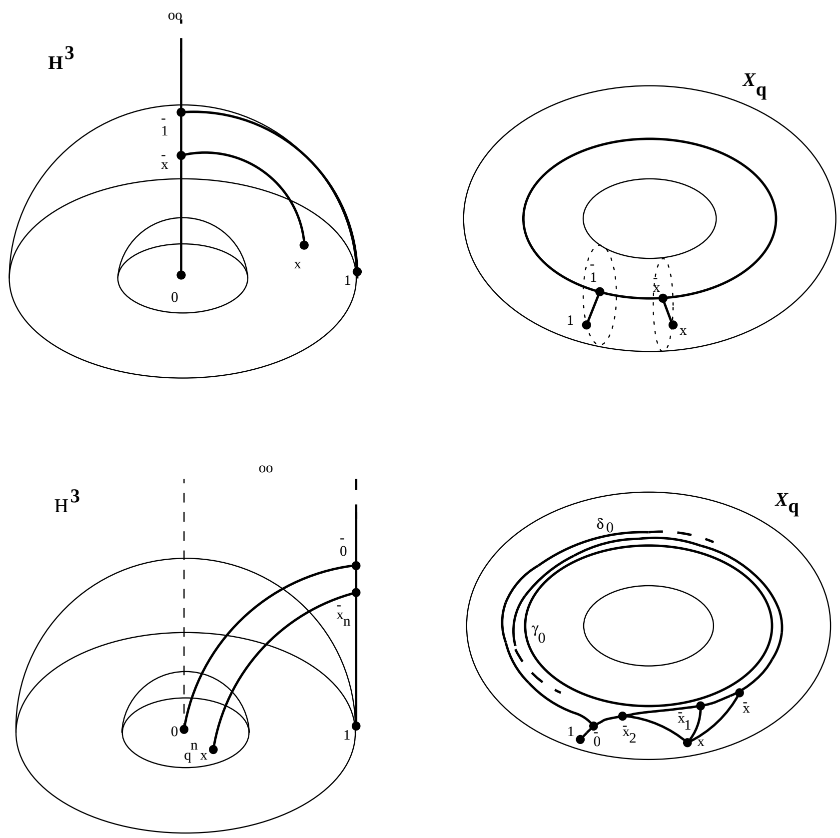
Arakelov Green function and BTZ black hole

Operator product expansion of path integral on the elliptic curve $X_q(\mathbb{C})$ (Alvarez-Gaume,Moore,Vafa Comm.Math.Phys. 106 1 (1986))

$$g(z,1) = \log\left(|q|^{B_2(\log|z|/\log|q|)/2}|1-z| \prod_{n=1}^{\infty} |1-q^n z|\ |1-q^n z^{-1}|\right)$$

in terms of geodesics (gravity on bulk space):

$$= -\frac{1}{2}\ell(\gamma_0)\, B_2\left(\frac{\ell_{\gamma_0}(\bar z, \bar 1)}{l(\gamma_0)}\right) + \sum_{n\geq 0} \ell_{\gamma_1}(\bar 0, \bar z_n) + \sum_{n\geq 1} \ell_{\gamma_1}(\bar 0, \tilde z_n).$$



$$\bar{x} = x * \{0, \infty\};\ \bar{z}_n = q^n z * \{1, \infty\},\ \tilde{z}_n = q^n z^{-1} * \{1, \infty\}$$

Higher genus case

Bosonic field propagator on algebraic curve $X_{\mathbb{C}}$ via

$$\omega_{(a)-(b)} := \nu_{(a)-(b)} - \sum_l X_l(a,b)\omega_{g_l},$$

differentials of the third kind with purely imaginary periods (Ferrari-Sobczyk, J.Math.Phys. 41 9 (2000))

All correlation functions:

$$G(z_1,\dots,z_m;w_1,\dots,w_\ell) = \sum_{j=1}^{m}\sum_{i=1}^{\ell} q_i\langle\phi(z_i,\bar{z}_i)\phi(w_j,\bar{w}_j)\rangle q'_j,$$

q_i = system of charges at positions z_i interacting with
q'_j = charges at positions w_j
obtained from basic correlator $G_\mu(a-b,z)$ (in terms of $\omega_{(a)-(b)}(z)$)

Schottky uniformization

$PSL(2,\mathbb{C})$ = orientation preserving isometries of $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$, 3-dim real hyperbolic space.

Schottky group: $\Gamma \subset PSL(2,\mathbb{C})$

- Γ is discrete, free group of rank $g \geq 1$
- The action of Γ on $\mathbb{H}^3$ extends to an action on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations
- Γ is purely loxodromic Kleinian group i.e.

$\forall$ generator $\gamma \in \Gamma$ $\exists$ $\{z^{\pm}(\gamma)\} \in \mathbb{P}^1(\mathbb{C})$ fixed points

$\mathbb{P}^1(\mathbb{C}) \supset \Lambda_\Gamma$ = limit set of Γ: accumulation pts. of Γ-orbits: Γ-invariant, totally disconnected (Cantor set for $g \geq 2$) closed subset of $\mathbb{P}^1(\mathbb{C})$

$\Omega_\Gamma := \mathbb{P}^1(\mathbb{C}) \smallsetminus \Lambda_\Gamma$ connected, non-simply connected Γ-invariant domain of discontinuity of Γ

$$X_{\mathbb{C}} = \Omega_\Gamma / \Gamma$$

Basis of differentials of the third kind as averages over the Schottky group: (base point $z_0 \in \Omega_\Gamma$, domain of discont) diff third kind

$$\nu_{(a)-(b)} := \sum_{\gamma \in \Gamma} d \log \langle a, b, \gamma z, \gamma z_0 \rangle$$

and diff first kind ($\{z^+(\gamma), z^-(\gamma)\}$ fixed points)

$$\omega_\gamma = \sum_{h \in C(|\gamma)} d \log \langle h z^+(\gamma), h z^-(\gamma), z, z_0 \rangle$$

(solve for $X_l(a,b)$ for purely imaginary periods)

Obtain explicit expression for the Green function:

$$g((a)-(b),(c)-(d)) = \sum_{h \in \Gamma} \log |\langle a, b, hc, hd \rangle|$$

$$- \sum_{\ell=1}^{g} X_\ell(a,b) \sum_{h \in S(g_\ell)} \log |\langle z^+(h), z^-(h), c, d \rangle|$$

$C(|\gamma) = \Gamma / \gamma^{\mathbb{Z}}$; $S(\gamma) =$ conjug.class

Green function and Krasnov black holes

Global quotients of AdS_{2+1} by a Schottky group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$; Euclidean signature: $\mathbb{H}^3/\Gamma$ hyperbolic handlebody, conformal boundary $X_{\mathbb{C}}$

Explicit bulk/boundary correspondence: each term in the Bosonic field propagator for $X_{\mathbb{C}}$ in terms of geodesics in Euclidean Krasnov black hole $\mathbb{H}^3/\Gamma$.

$$-\sum_{h\in\Gamma} \mathrm{ordist}(a * \{hc, hd\}, b * \{hc, hd\})$$

$$+\sum_{\ell=1}^{g} X_\ell(a,b) \sum_{h\in S(g_\ell)} \mathrm{ordist}(z^+(h) * \{c,d\}, z^-(h) * \{c,d\}).$$

Coefficients $X_\ell(a,b)$ also in terms of geodesic propagators